Chapter 3 Notes

The Derivative

We now discuss the idea of a derivative which is fundamental to calculus.

1 The Definition

Recall that the slope of the tangent line to a curve $f(x)$ at $x = a$ is defined by

$$m_{\tan} = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

and given a position function $s(t)$ we can fine the velocity of a particle at $t = a$ by

$$v(a) = \lim_{b \to a} \frac{s(b) - s(a)}{b - a}$$

These are two examples of derivatives of a function.

Definition 1. Let a function f be defined in an open interval containing a point a. The **derivative** of f at a is the limit

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

if it exists. If $f'(a)$ exists, we say that f is **differentiable** at a.

The following is a graphical interpretation of the derivative.

We can interpret the derivative as the slope of the tangent line of the graph of a function, or as the rate of change of a function with respect to its independent variable.

Example 1. If $f(x) = \sqrt{x}$ find $f'(a)$ and the equation for the tangent line to f at a.

Example 2. Find the derivative of a general quadratic polynomial, $f(x) = ax^2 + bx + c$ for $\{a, b, c\} \in \mathbb{R}$ at $x = d$.

1.1 Differentiability and Continuity

Theorem 2. *If a function f is differentiable at a, then f is continuous at a.*

Proof:

If f is differentiable at a, then a is an interior point of the domain of f. Thus it suffices to show that

$$\lim_{x \to a} [f(x) - f(a)] = 0$$

to show continuity of f. Since

$$\lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot (x - a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} (x - a) = f'(a) \cdot 0 = 0$$

we find

$$\lim_{x \to a} [f(x) - f(a)] = 0$$

and we are done.

□

As a result of this theorem, we find that a function f is not differentiable at some point a, if it has a discontinuity at a or the graph of the function has a "sharp corner" at a.

1.2 The Derived Function

In the definition of the derivative at a, setting $b = a + h$ and letting $b \to a$ iff $h \to 0$ we find an equivalent definition of the derivative is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If a is any point in the domain of f then we may replace it by a variable x from which we find a definition for the **derivative of** f to be

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

If $f'(x)$ exits, we say f is **differentiable** at x. If $f'(x)$ exists for all x in a set S, we say f is **differentiable on** S. If $f'(x)$ exists for all x in the domain of f, we say simply that f is **differentiable**.

Example 3. Find the derivatives of the following functions:

$$f(x) = \sqrt{x} \quad g(x) = ax^2 + bx + c$$

1.3 Notations for the Derivative

There are many ways to notate the derivative of a function f. $f'(x)$ is the most often used. If we set $y = f(x)$, y' is another notation for the derivative. We can denote the derivative of f at a point a by $f'(a)$.

We will also use **Leibnitz notation** of the form

$$\frac{df}{dx} \quad \text{or} \quad \frac{dy}{dx}$$

often since we will find it has particular advantages later.

We denote the derivative at a point a by

$$\left.\frac{df}{dx}\right|_{x=a}$$

Next there is Newtonian dot notation, where the derivative of a function with respect to its argument is written $\dot{f}$. This notation is often used when the independent variable represents time.

1.4 Higher Order Derivatives

Given a function f, we can differentiate it to obtain a function f'. We can again differentiate f' to obtain a new function f'', which in Leibnitz notation is denoted

$$\frac{d^2x}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

Acceleration is an example of this. We will later see that the second derivative of a function is related to the **concavity** of the graph of that function.

We can also take higher derivatives which we represent by

$f''', f^{(4)}, f^{(5)}, \ldots f^{(n)}$ or in Leibnitz notation

$$\frac{d^2x}{dx^2}, \frac{d^3x}{dx^3}, \ldots, \frac{d^nx}{dx^n}$$

If $f^{(n)}$ exists we way f is n times differentiable and we denote this by $f \in C^n$. If $f^{(n)}$ exists for all positive n we say f is infinitely differentiable and denote this by $f \in C^\infty$. If f is just continuous but not differentiable, we say $f \in C^0$.

2 Derivatives of Rational Functions

Up to this point, computing derivatives has shown to be a tedious task. We will develop some formulas in this section to reduce the amount of calculation necessary to find derivatives of functions.

2.1 Derivatives of Constant, Linear, and Power Functions

Theorem 3. *For* $c \in \mathbb{R}$

$$\frac{dc}{dx} = 0$$

Proof:

□

Theorem 4. *For* $\{m, b\} \in \mathbb{R}$*, the function* $f(x) = mx + b$ *is differentiable and*

$$\frac{df}{dx} = m$$

Proof: We proved this in this previous section when we computed the derivative of a quadratic function.

□

Theorem 5. *(The Power Rule) If* n *is a positive integer, then the power function* $f(x) = x^n$ *is differentiable and its derivative is* $f'(x) = nx^{n-1}$ *or*

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Proof:

Example 4. Find the derivatives of the following

$$f(x) = 4x^2 \quad g(x) = x^{10} \quad h(a) = ax^n$$

2.2 Derivatives of Sums and Differences

Theorem 6. *(The Sum Rule or Linearity Rule) If* $\{u(x), v(x)\} \in C^1$ *then*

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

Proof:

Example 5. Find the derivative of

$$f(x) = 4x^2 + 3x - 7$$

Theorem 7. *(Constant Multiple Rule) If $c \in \mathbb{R}$ and $f(x) \in C^1$, then*

$$\frac{d}{dx}(cf(x)) = c\frac{df}{dx}$$

Proof: Exercise

□

2.3 Derivatives of Products and Quotients

Theorem 8. *If $\{u(x), v(x)\} \in C^1$ then*

$$\frac{d}{dx}(uv) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

Proof:

The product rule can be remembered by the phrase “first times the derivative of the second plus the second times the derivative of the first”.

Example 6. Find the derivative of the function $f(x) = (3x^2)(x-1)$

Theorem 9. *(The Reciprocal Rule) For $v(x) \in C^1$, when $1/v(x)$ is defined*

$$\frac{d}{dx}\left(\frac{1}{v}\right) = -\frac{1}{v^2}\frac{dv}{dx}$$

Proof:

Using the reciprocal rule, the power rule may be extended to negative integers.

Review example 54 on page 182 before attempting the homework.

Theorem 10. *(The Quotient Rule) If $\{u(x), v(x)\} \in C^1$ then whenever u/v is defined,*

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

Proof: The proof combines both the product and reciprocal rules.

The quotient rule can be remembered by the phase, "bottom times the derivative of the top minus the top times the derivative of the bottom all over the bottom squared."

Example 7. Find the derivative of $f(x) = \frac{x^2+4}{x^3-2}$

3 Derivatives of Exponential and Trigonometric Functions

Up to this point we can only differentiate functions that are sums or differences of polynomials. We will now extend our methods to other types of functions.

3.1 Derivatives of Exponential Functions

Consider the exponential function

$$f(x) = a^x$$

where $a > 0$ and $a \neq 1$. We use the definition of the derivative to compute

$$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} a^x \cdot \frac{a^h - 1}{h} = a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h}$$

We now investigate the limit of the second factor.

Assume the second factor is 1, so that we have an exponential function which is its own derivative. Then for $h \approx 0$

$$\frac{a^h - 1}{h} \approx 1 \Rightarrow a \approx (1 + h)^{1/h}$$

so we predict that

$$a = \lim_{h \to 0} (1 + h)^{1/h} = e \approx 2.71828$$

The above is not a rigorous argument, however it does imply a true theorem:

Theorem 11.

$$\frac{d}{dx}(e^x) = e^x$$

Proof: We will prove this later.

□

More generally the following also holds,

Theorem 12.

$$\frac{d}{dx}(a^x) = a^x \ln a$$

Proof: We will also prove this later.

□

3.2 Derivatives of Trigonometric Functions

Will will use the limits

$$\lim_{h \to 0} \frac{\sin h}{h} = 1 \qquad \lim_{h \to 0} \frac{\cos h - 1}{h} = 0$$

in the following.

Theorem 13.

$$\frac{d}{dx}(\sin x) = \cos x$$

Proof: The proof is given on page 189

□

Theorem 14.

$$\frac{d}{dx}(\cos x) = -\sin x$$

Proof:

Theorem 15.

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

Proof:

In proving these identities it is often helpful to remember that

$$\tan x = \frac{\sin x}{\cos x} \quad \cot x = \frac{\cos x}{\sin x} \quad \sec x = \frac{1}{\cos x} \quad \csc x = \frac{1}{\sin x}$$

We will derive one of these and the rest will be exercises.

Example 8. Find the derivatives of the following

$$f(x) = e^{ax} \sin bx \quad g(x) = \frac{x \sin x}{1 - \cos x} \quad h(x) = e^{ax} b^x$$

4 The Chain Rule

We will now see how to differentiate compositions of functions via the **chain rule**.

Theorem 16. *(Chain Rule)*

$$\frac{d}{dx}[g(f(x))] = (g \circ f)(x) = g'(f(x)) \cdot f'(x) = \frac{dg}{df} \cdot \frac{df}{dx}$$

Proof: The proof is semi-long and can be found on pages 195-196 of the text.

The chain rule can be remembered by the phrase "differentiate the outside function leaving the inside alone and multiply the result by the derivative of the inside."

Make sure that you have all of the differentiation rules listed in this chapter **committed to memory**.

We now demonstrate the chain rule via several examples:

Example 9. Differentiate $h(x) = \tan ax$,$s(x) = e^{-4x}$, and $r(x) = \sin^2 x$

Example 10. If $y = u^4 - 4u^2$ and $u = x^2 - e^x$ find $dy/dx|_{x=3}$

Example 11. Differentiate $f(x) = ae^{bx} \cos^3 cx$

5 Implicit Differentiation

Consider the equation for the unit circle

$$x^2 + y^2 = 1$$

This equation is not a function. Why? Implicit differentiation will allow us to find the derivative of the graph of this equation.

5.1 Implicit Functions

Definition 17. If f is a function of x, then $f(x)$ is an **explicit** function of x.

Definition 18. Suppose that $g(x, y)$ is an expression in the variables x and y. If there is a function $f(x)$ such that $g(x, f(x)) = 0$ for every x in the domain of f, then the equation $g(x, y) = 0$ is said to define y as a function of x **implicitly**.

Example 12. Class Examples

5.2 Implicit Differentiation

We will demonstrate implicit differentiation with the following example.

Consider the equation

$$5x^2 - 2y^2 + 3 = 0$$

where y is a function of x. Differentiation with respect to x, we find

$$\frac{d}{dx}(5x^2 - 2y^2 + 3) = 0$$
$$10x - 4y\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$ we find

$$\frac{dy}{dx} = \frac{5x}{2y}$$

If we wish to make this depend solely on x, we may solve the initial equation to obtain

$$y = \pm\sqrt{\frac{3 + 5x^2}{2}}$$

and substitute the result into the above equation. Note that y **is not a function**, and the derivative is only defined explicitly piecewise.

In general, when implicitly differentiating, differentiate both sides of a given equation using the chain rule for any term that involves y (the dependent variable). Then solve for dy/dx.

Example 13. Find dy/dx for

$$x^2 + y^{-2} = 4 \quad \text{and} \quad xy = \sin(x + y)$$

Example 14. Find the first and second derivatives of

$$x^2 - y^2 = a$$

Example 15. Find the line tangent to the curve $x^3 + 4y^2 = 5$ at the point $(1, 1)$

5.3 Derivatives of Inverse Functions

Suppose we are given an explicit function $y = f(x)$. Then interchanging x and y, we will get a new function $x = f(y)$ which defines y implicitly as an inverse function, assuming such exists. So suppose f is differentiable and has an inverse function f^{-1}. If $y = f^{-1}(x)$ then $f(y) = x$. Differentiating we have

$$f'(y)\frac{dy}{dx} = 1 \to \frac{dy}{dx} = \frac{1}{f'(y)}$$

Since $y = f^{-1}(x)$, we have

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$$

This is part of the proof of the following theorem

Theorem 19. *If f is a differentiable function with inverse function f^{-1}, then f^{-1} is differentiable for all x such that $f'(f^{-1}(x)) \neq 0$, and*

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$$

Using the inverse function rule, one can extend the power rule to find

$$\frac{d}{dx}x^{r/s} = \frac{r}{s}x^{r/s-1}$$

as is done on page 207 of the text.

Example 16. Find dy/dx for the following

$$y = x^{1/5} \quad y = \sqrt{x^3 + 2} \quad y = |x| \quad y = |f(x)|$$

Theorem 20.

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{x^2 + 1}$$

Proof: Before we prove these three statements, note there are generalizations in the text of each given in (3.60)-(3.64) on page 210.