

Calculus Solutions: Chapter 6.7

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Integrate:

1b. $\int_0^3 (x-2)^{4/3} dx$

Solution:

Let $u = x - 2$. Then $du = dx$ and we find

$$\int_0^3 (x-2)^{4/3} dx = \int_0^3 u^{4/3} du = \frac{3}{7} u^{7/3} \Big|_{x=0}^3 = \frac{3}{7} (x-2)^{4/3} \Big|_0^3 = \frac{3}{7}$$

□

1d. $\int_{-2}^2 (4-x)^{1/3} dx$

Solution:

Let $u = 4 - x$. Then $du = -dx$ and we find

$$\int_{-2}^2 (4-x)^{1/3} dx = - \int_{-2}^2 u^{1/3} du = \frac{3}{4} u^{4/3} \Big|_{x=-2}^2 = -\frac{3}{4} (4-x)^{4/3} \Big|_{-2}^2 = \frac{9 \cdot 3^{1/3} - 3}{2^{2/3}}$$

□

1f. $\int 3x\sqrt{x^2+9} dx$

Solution:

Let $u = x^2 + 9$. Then $du = 2x dx \rightarrow dx = \frac{du}{2x}$. Substituting into the original integral we find

$$\int 3x\sqrt{x^2+9} dx = \frac{3}{2} \int u^{1/2} = u^{3/2} + C = (x^2+9)^{3/2} + C$$

□

1h. $\int \frac{x}{x^2+1} dx$

Solution:

Let $u = x^2 + 1$. Then $du = 2x dx \rightarrow dx = \frac{du}{2x}$. Substituting into the original integral we find

$$\int \frac{x}{x^2+1} dx = \frac{1}{2} \int u^{-1} du = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |x^2+1| + C$$

□

1j. $\int \frac{\sqrt{1+x^{3/5}}}{x^{2/5}} dx$

Solution:

Let $u = 1 + x^{3/5}$. Then $du = \frac{3}{5}x^{-2/5}dx \rightarrow dx = \frac{5}{3}x^{2/5}$. Substituting into the original integral we find

$$\int \frac{\sqrt{1+x^{3/5}}}{x^{2/5}} dx = \frac{5}{3} \int u^{1/2} du = \frac{10}{9} u^{3/2} + C = \frac{10}{9} (1+x^{3/5})^{3/2} + C$$

□

Integrate:

2b. $\int \sin x \cos^2 x dx$

Solution:

Let $u = \cos x$. Then $du = -\sin x dx \rightarrow dx = -\csc x du$. Substituting into the integral we find

$$\int \sin x \cos^2 x dx = - \int u^2 du = -\frac{1}{3} u^3 + C = -\frac{1}{3} \cos^3 x + C$$

□

2d. $\int_0^{\pi/4} \tan x \sec^2 x dx$

Solution:

Let $u = \tan x$. Then $du = \sec^2 x dx \rightarrow dx = \cos^2 x du$. Substituting into the integral we find

$$\int_0^{\pi/4} \tan x \sec^2 x dx = \int_0^{\pi/4} u du = \frac{1}{2} u^2 \Big|_{x=0}^{\pi/4} = \frac{1}{2} \tan^2 x \Big|_0^{\pi/4} = \frac{1}{2}$$

□

2f. $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$

Solution:

Let $u = \sqrt{x}$. Then $du = \frac{1}{2}x^{-1/2}dx \rightarrow dx = 2x^{1/2}du$. Substituting into the integral we find

$$\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx = 2 \int \cos u du = 2 \sin u + C = 2 \sin \sqrt{x} + C$$

□

2h. $\int \tan x dx$

Solution:

Applying the definition of tangent and the Fundamental Theorem we find

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = \int \frac{d}{dx}(-\ln |\cos x|) dx = \ln |\cos x| + C$$

□

2j. $\int \sin(ax) dx$

Solution:

$$\int \sin(ax) dx = \int \frac{d}{dx} \left(-\frac{1}{a} \cos(ax) \right) dx = -\frac{1}{a} \cos ax + C$$

□

Integrate:

3b. $\int_0^3 e^{2x-1} dx$

Solution:

$$\int_0^3 e^{2x-1} dx = e \int_0^3 e^{2x} dx = \frac{e}{2} e^{2x} \Big|_0^3 = \frac{e}{2} (e^6 - 1)$$

□

3d. $\int x e^{-x^2/3} dx$

Solution:

Let $u = -\frac{x^2}{3}$. Then $du = -\frac{2}{3}x dx \rightarrow dx = -\frac{3}{2x} du$. Substituting into the integral we find

$$\int x e^{-x^2/3} dx = -\frac{3}{2} \int e^u du = -\frac{3}{2} e^u + C = -\frac{3}{2} e^{-x^2/3} + C$$

□

3f. $\int 10^{-3x} dx$

Solution:

$$\int 10^{-3x} dx = \int \frac{d}{dx} \left(-\frac{10^{-3x}}{3 \ln 10} \right) = -\frac{10^{-3x}}{3 \ln 10} + C$$

□

3g. $\int_1^2 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

Solution:

Let $u = \sqrt{x}$. Then $du = \frac{1}{2}x^{-1/2}dx \rightarrow dx = 2x^{1/2}du$. Substituting into the integral we find

$$\int_1^2 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int_1^2 e^u du = 2e^u \Big|_{x=1}^2 = 2e^{\sqrt{x}} \Big|_1^2 = 2(e^{\sqrt{2}} - e)$$

□

3h. $\int \frac{e^x}{1+e^x} dx$

Solution:

Let $u = 1 + e^x$. Then $du = e^x dx \rightarrow dx = e^{-x} du$. Substituting into the integral we find

$$\int \frac{e^x}{1+e^x} dx = \int u^{-1} du = \ln |u| + C = \ln |1 + e^x| + C$$

□

3j. $\int \frac{1}{x \ln x} dx$

Solution:

Let $u = \ln x$. Then $du = \frac{1}{x} dx \rightarrow dx = x du$. Substituting into the integral we find

$$\int \frac{1}{x \ln x} dx = \int u^{-1} du = \ln |u| = \ln |\ln x|$$

□

3l. $\int \frac{e^x}{1+e^{2x}} dx$

Solution:

$$\int \frac{e^x}{1+e^{2x}} dx = \int \frac{d}{dx} (\tan^{-1} e^x) dx = \tan^{-1} e^x + C$$

□

3o. $\int \frac{\ln(ax)}{x} dx$

Solution:

Let $u = \ln(ax)$. Then $du = \frac{1}{x} dx \rightarrow dx = x du$. Substituting into the integral we find

$$\int \frac{\ln(ax)}{x} dx = \int u du = \frac{1}{2} u^2 + C = \frac{1}{2} (\ln(ax))^2 + C$$

□

Integrate:

4a. $\int \frac{x}{\sqrt{x-4}} dx$

Solution:

Let $u = x - 4$. Then $du = dx$ and $x = u + 4$. Substituting into the original integral we find

$$\begin{aligned}\int \frac{x}{\sqrt{x-4}} dx &= \int \frac{u+4}{u^{1/2}} du = \int (u^{1/2} + 4u^{-1/2}) du = \frac{2}{3}u^{3/2} + 8u^{1/2} + C \\ &= \frac{2}{3}(x-4)^{3/2} + 8(x-4)^{1/2} + C\end{aligned}$$

□

4b. $\int 3x\sqrt{x+5} dx$

Solution:

Let $u = x + 5$. Then $du = dx$ and $x = u - 5$. Substituting into the original integral we find

$$\begin{aligned}\int 3x\sqrt{x+5} dx &= \int 3(u-5)u^{1/2} du = 3 \int (u^{3/2} - 5u^{1/2}) du = \frac{6}{5}u^{5/2} - 10u^{3/2} + C \\ &= \frac{6}{5}(x+5)^{5/2} - 10(x+5)^{3/2} + C\end{aligned}$$

□

4d. $\int \frac{x}{x+5} dx$

Solution:

Let $u = x + 5$. Then $du = dx$ and $x = u - 5$. Substituting into the original integral we find

$$\int \frac{x}{x+5} dx = \int \frac{u-5}{u} du = \int 1 - \frac{5}{u} du = u - 5 \ln |u| + C = (x+5) + \ln |x+5| + C$$

□

4f. $\int x^3\sqrt{9-x^2} dx$

Solution:

Let $u = 9 - x^2$. Then $du = -2x dx \rightarrow dx = -\frac{du}{2x}$ and $x^2 = 9 - u$. Substituting into the original integral we find

$$\int x^3\sqrt{9-x^2} dx = -\frac{1}{2} \int x^2 u^{1/2} du = -\frac{1}{2} \int (9-u)u^{1/2} du$$

$$\begin{aligned}
&= -\frac{1}{2} \int 9u^{1/2} - u^{3/2} du = -\frac{1}{2} \left(6u^{3/2} - \frac{2}{5}u^{5/2} \right) + C \\
&= -\frac{1}{2} \left(6(9-x^2)^{3/2} - \frac{2}{5}(9-x^2)^{5/2} \right) + C
\end{aligned}$$

□

Give the general solution of the differential equation.

5a. $\frac{y}{x} \frac{dy}{dx} = \sqrt{1-x^2}$

Solution:

Separating variables, we find

$$y dy = \frac{\sqrt{1-x^2}}{x} dx$$

Integrating we have

$$\frac{1}{2}y^2 = -\frac{x^2}{2} + \ln|x| + C_1$$

Solving for y we conclude

$$y = \pm \sqrt{-x^2 + 2 \ln|x| + C}$$

□

5b. $\frac{dy}{dx} = \frac{x}{y^2} \cos(x^2 + 1)$

Solution:

Separating variables we find

$$y^2 dy = x \cos(x^2 + 1) dx$$

Integrating we find

$$\frac{1}{3}y^3 = \frac{1}{2} \sin(1 + x^2) + C_1$$

Solving for y we conclude

$$y = \left(\frac{3}{2} \sin(1 + x^2) + C \right)^{1/3}$$

□

Give the general solution of the differential equation and the particular solution satisfying the given boundary conditions.

6a. $\sec x \frac{dy}{dx} = y^2 e^{\sin x}$, $y(0) = 2$

Solution:

Separating variables we find

$$\frac{dy}{y^2} = \cos x e^{\sin x} dx$$

Integrating we find

$$\frac{1}{3}y^3 = e^{\sin x} + C_1$$

Solving for y we have

$$y = \left(3e^{\sin x} + C\right)^{1/3}$$

Since $y(0) = 2$ we find

$$2 = (3 + c)^{1/3} \rightarrow c = 5$$

Thus the solution to the initial value problem is

$$y = \left(3e^{\sin x} + 5\right)$$

□

6b. $\frac{y^2}{\tan^{-1} x} \frac{dy}{dx} = \frac{4}{1+x^2}, y(0) = 0$

Solution:

Separating variables we find

$$y^2 dy = \frac{4 \tan^{-1} x}{1+x^2} dx$$

Integrating we find

$$\frac{1}{3}y^3 = 2(\tan^{-1} x)^2 + C_1$$

Solving for y we find

$$y = [6(\tan^{-1} x)^2 + C]^{1/3}$$

Since $y(0) = 0$, we find $c = 0$. Thus the solution to the initial value problem is

$$y = [6(\tan^{-1} x)]^{1/3}$$

□

Derive the formulas, either directly by substitution or indirectly by differentiating the right-hand side.

We differentiate the right side of each equation in the following and note the result is exactly the integrand of the left side.

7b. $\int \frac{1}{ax+b} dx = \frac{1}{a} \ln |ax + b| + C$

Solution:

$$\frac{d}{dx} \left(\frac{1}{a} \ln |ax + b| \right) = \frac{1}{a} \cdot \frac{a}{ax + b} = \frac{1}{ax + b}$$

□

7d. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \frac{x}{a} + C$ if $a > 0$

Solution:

$$\frac{d}{dx} \left(\sin^{-1} \frac{x}{a} \right) = \frac{1}{a} \cdot \frac{1}{\sqrt{1 - (x/a)^2}} = \frac{1}{a} \cdot \frac{a}{\sqrt{a^2 - x^2}} = \frac{1}{\sqrt{a^2 - x^2}}$$

□

7f. $\int \cot(ax) dx = \frac{1}{a} \ln |\sin(ax)| + C$

Solution:

$$\frac{d}{dx} \left(\frac{1}{a} \ln |\sin(ax)| \right) = \frac{1}{a} \cdot \frac{a \cos(ax)}{\sin(ax)} = \cot(ax)$$

□

13. Now that we know

$$\frac{d}{dx}(e^x) = e^x$$

we can prove

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

Do so. (Now all the differentiation and integration formulas we have derived have been proven formally).

Solution:

We apply L' Hopital's rule to find

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \lim_{x \rightarrow 0} e^x = e^0 = 1$$

□