

Calculus Solutions: Chapter 6.5

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Integrate:

1b. $\int_{-1}^1 x^{10} dx$

Solution:

$$\int_{-1}^1 x^{10} = \frac{1}{11} x^{11} \Big|_{-1}^1 = \frac{2}{11}$$

□

1d. $\int_1^8 x^{4/3} dx$

Solution:

$$\int_1^8 x^{4/3} dx = \frac{3}{7} x^{7/3} \Big|_1^8 = \frac{3}{7} (8^{7/3} - 1)$$

□

1f. $\int_{-1}^{-2} 2x^{-5} dx$

Solution:

$$2 \int_{-1}^{-2} 2x^{-5} dx = 8x^{-4} \Big|_{x=-1}^{-2} = 8 \left(\frac{1}{16} - 1 \right) = -\frac{15}{2}$$

□

1h. $\int_{-2}^{-1} x^{-2/3} dx$

Solution:

$$\int_{-2}^{-1} x^{-2/3} dx = 3 \int_{-2}^{-1} x^{1/3} dx = 3 \left(\frac{3}{4} x^{4/3} \right) \Big|_{-2}^{-1} = \frac{9}{4} (1 - 2^{4/3})$$

□

Integrate:

2b. $\int_0^{\pi/3} \sin x dx$

Solution:

$$\int_0^{\pi/3} \sin x dx = -\cos x \Big|_0^{\pi/3} = 1 - \frac{1}{2} = \frac{1}{2}$$

□

2d. $\int_0^{\pi/4} (\cos x - \sin x) dx$

Solution:

$$\int_0^{\pi/4} (\cos x - \sin x) dx = \sin x + \cos x \Big|_0^{\pi/4} = \sqrt{2} - 1$$

□

2f. $\int_{\pi/4}^{\pi/3} \csc x \cot x dx$

Solution:

$$\int_{\pi/4}^{\pi/3} \csc x \cot x dx = -\int_{\pi/4}^{\pi/3} \frac{d}{dx}(\csc x) dx = \csc x \Big|_{\pi/3}^{\pi/4} = \frac{2\sqrt{3}}{3} - \sqrt{2}$$

□

2h. $\int_{\pi/6}^{\pi/4} \csc^2 x dx$

Solution:

$$\int_{\pi/6}^{\pi/4} \csc^2 x dx = -\cot x \Big|_{\pi/6}^{\pi/4} = \cot x \Big|_{\pi/4}^{\pi/6} = 1 - \sqrt{3}$$

□

Integrate:

3b. $\int_0^1 3^x dx$

Solution:

$$\int_0^1 3^x = \frac{3^x}{\ln 3} \Big|_0^1 = \frac{2}{\ln 3}$$

□

3d. $\int_0^{0.5} 2^{-x} dx$

Solution:

$$\int_0^{0.5} 2^{-x} dx = -\frac{2^{-x}}{\ln 2} \Big|_0^{0.5} = \frac{2 - \sqrt{2}}{\ln 4}$$

□

3f. $\int_0^1 \frac{1}{x^2+1} dx$

Solution:

$$\int_0^1 \frac{1}{x^2+1} dx = \tan^{-1} x \Big|_0^1 = \frac{\pi}{4}$$

□

3h. $\int_1^3 \frac{x^2+1}{x} dx$

Solution:

$$\int_1^3 \frac{x^2+1}{x} dx = \int_1^3 x + \frac{1}{x} dx = \frac{1}{2}x^2 + \ln|x| \Big|_1^3 = 4 + \ln 3$$

□

4b. Find $\frac{dA}{dx}$ if $A(x) = \int_1^x (t^3 + t^{-2}) dt$

Solution:

Applying the Fundamental theorem we find

$$\frac{dA(x)}{dx} = \int_1^x (t^3 + t^{-2}) dt = x^3 + x^{-2}$$

□

6d. Find $\frac{dA}{dx}$ if $A(x) = \int_{x^2}^{\sqrt{x}} \sin t^2 dt$

Solution:

We set $f(t) = \sin t^2$ and rewrite $A(x)$ as

$$A(x) = \int_0^{\sqrt{x}} \sin t^2 dt - \int_0^{x^2} \sin t^2 dt = \int_0^{\sqrt{x}} f(t) dt - \int_0^{x^2} f(t) dt$$

Applying the Fundamental theorem we find

$$\frac{dA(x)}{dx} = f(\sqrt{x}) - f(x^2) = \sin x + \sin x^4$$

□

9. If an object has velocity $v(t)$ and position s_1 at time t_1 , explain why its position at any time T is $s(T) = s_1 + \int_{t_1}^T v(t) dt$

Solution:

Given an average velocity and a time interval, we can compute the distance traveled during the interval by multiplying the two. The integral is doing just this, an infinity of infinitesimal intervals dt and velocities $v(t)$. We add on the initial position to find the position of the object relative to our desired origin.

□

10b. Find position $s(T)$, given the initial position $s(a) = 12$, $a = 2$ and velocity $v(t) = 4t - t^2$, $T = 10$

Solution:

Applying the above formula in problem 9 we have

$$s(T) = 12 + \int_2^T (4t - t^2) dt = \frac{T^2}{3}(6 - T) + 12$$

□

12. Given, for $x \in [0, \infty)$

$$F(x) = \int_0^{x^3} \sqrt{1+t^2} dt$$

a) show that F^{-1} exists

Solution:

From problem 5 we calculate

$$\frac{dF}{dx} = 3x^2 \sqrt{1+x^6} > 0 \quad \text{for } x \in [0, \infty)$$

Thus F is monotonically increasing on $[0, \infty)$ and therefore has an inverse by the horizontal line test.

□

b) show that $F(a) = 1$ for some $a > 0$

Solution:

We note that $F(0) = 0$ and $F(4) \approx 9.29357$. Since $F(x)$ is continuous and monotonically increasing, there must be some a such that $F(a) = 1$ by the intermediate value theorem.

□

c) find $\left. \frac{d}{dx} F^{-1}(x) \right|_{x=1}$

Solution:

Applying the derivative inverse rule on page 206, we find

$$\frac{d}{dx} F^{-1} = \frac{1}{F'(F^{-1}(x))}$$

Plotting $F(x)$ and recalling that the graph of $F^{-1}(x)$ is the reflection of $F(x)$ about the $y = x$ graph, we estimate $F^{-1}(1) = .95$. Thus applying the previous formula we find:

$$\left. \frac{d}{dx} F^{-1} \right|_{x=1} \approx \frac{1}{F'(.95)} \approx 3.5664$$

13. Show that if f is continuous and increasing on $[a, b]$, then the function $A(x) = \int_a^x f(t)dt, x \in [a, b]$, is concave upward. If f is decreasing, show that A is concave downward.

Solution:

We find that $A'(x) = f(x)$. If f is increasing on $[a, b]$ then $f'(x) > 0$ on $[a, b]$. Thus we find $A''(x) > 0$ on $[a, b]$ and hence A is concave upward. If f is decreasing on $[a, b]$ then $f'(x) < 0$ on $[a, b]$. Thus we find $A''(x) < 0$ on $[a, b]$ and hence A is concave downward.

□

16. For the graph shown in Figure 6.26, sketch the graph of $A(x) = \int_1^x f(t)dt$ for $x \in [1, c]$.

Solution:

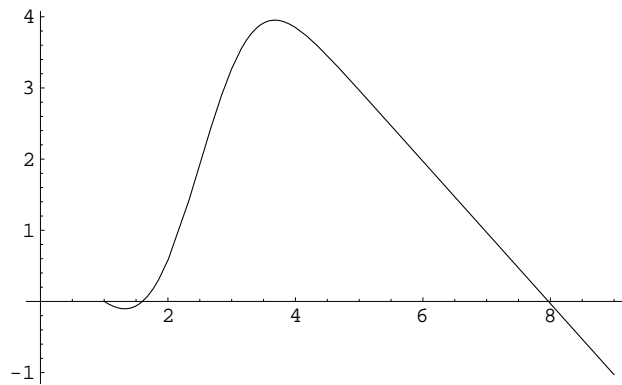


Figure 1: $A(x) = \int_1^x f(t)dt$ where $f(t) = 4e^{-(t-2.5)} - 1$ $x \in [0, 10]$

□