Calculus Solutions: Chapter 6.4

Aaron Peterson, Stephen Taylor

Use $L_f(n)$, $R_f(n)$, $T_f(n)$, $M_f(n)$, and $SR_f(2n)$ to estimate each of the following definite integrals for the given number n.

1b. $\int_0^\pi \sin x dx$, $n = 4$

Solution:

$$L_f(4) = R_f(4) = T_f(4) \approx 2.02093$$
$$M_f(4) \approx 2.05324$$
$$SR_f(2n) = 2.04247$$

□

1d. $\int_0^1 [1/(x^2+1)]dx$, $n = 10$

Solution:
The integral's value is $\pi/4$. Your approximations should be close to this.

□

1f. $\int_0^{\pi/4} \sin^3 x dx$, $n = 6$

Solution:
The integral's value is $\frac{8-5\sqrt{2}}{12}$. You approximations should be close to this.

□

2. Use $L_f(n)$ and $R_f(n)$ to estimate $\int_0^{\pi/4} \tan x dx$ for $n = 4$, $n = 20$, and $n = 100$. Explain how accurate the last estimates are.

Solution:
The integral's value is $\frac{\ln 2}{2}$. Your sums should be near this value.

□

3. Use $T_f(n)$ and $M_f(n)$ to estimate $\int_0^{\pi/4} \tan x dx$ for $n = 4$, $n = 20$, $n = 100$. How accurate do you think the last estimates are?

Solution:
The integral's value is $\frac{\ln 2}{2}$. Your sums should be near this value.

□

4. Use $SR_f(n)$ to estimate $\int_0^{\pi/4} \tan x dx$ for $n = 4, n = 20$ and $n = 100$. How accurate do you think the last estimate is?

Solution: The integral's value is $\frac{\ln 2}{2}$. Your sums should be near this value.

□

Use a method discussed in this section to estimate the given integral correct to within $5 \cdot 10^7$.

5b. $\int_0^1 1/(x^2+1)dx$

Solution:

$$\int_0^1 1/(x^2+1)dx \approx 0.785398163$$

□

5d. $\int_0^{0.6} e^{-x^2} dx$

Solution:

$$\int_0^{0.6} e^{-x^2} \approx 0.535154$$

□

5f. $\int_1^2 \frac{\sin x}{x} dx$

Solution:

$$\int_1^2 \frac{\sin x}{x} dx \approx 0.659329906$$

□

5h. $\int_1^2 \cos(\ln x)dx$

Solution:

$$\int_1^2 \cos(\ln x)dx \approx 0.908200178$$

□

6. A certain function f is increasing and concave downward over the interval $[a, b]$. The definite integral $\int_a^b f(x)dx$ has value I. The definite integral is estimated by $L_f(n)$, $R_f(n)$, $T_f(n)$, and $M_f(n)$. Arrange the number I and these four estimates in order from smallest to largest.

Solution:
Since f is increasing we know the left hand sum will be less than the right hand sum. Since f is concave downward we know the trapezoid sum will underestimate the integral and the midpoint sum will overestimate it. Noting both of these sums converge faster then the left and right sums, we find the relationship:

$$L_f(n) < T_f(n) < M_f(n) < R_f(n)$$

□

8. A certain definite integrals was estimated by $L_f(n)$, $T_f(n)$, $M_f(n)$, and $SR_f(2n)$ for values of n. These results are tabulated below, but the headings have been left off the columns.

a) Which column is which and why?

Solution:
The third column is $L_f(n)$ since it has the slowest convergence rate, and the fourth column is $SR_f(2n)$ since it has the fastest converging rate. The first column is $M_f(n)$ since it converges faster than the second which must be $T_f(n)$.

□

b) Assuming that the integrand of the definite integral involved has no extrema or inflection points over the interval of integration, describe the monotonicity and concavity of the integrand.

Solution:
Since the lower sum is decreasing we know the function must be increasing. Since the midpoint approximation is increasing, the curve must be concave up.

□

Considering the error estimates given in this section, for what type of integrand would you expect the following estimates to be exact?

14b. $R_f(n)$

Solution:
For any constant integrand, this sum will be exact.

□

14d. $M_f(n)$

Solution:
For any linear integrand, this sum will be exact.

□

15. Show that the error estimate (6.28) cannot be improved by showing that the error in estimating $\int_a^b Bxdx$ by $L_f(n)$ is exactly

$$\frac{B(b-a)^2}{2n}$$

Solution:
From the definition we find

$$L_f(n) = B\sum_{k=0}^{n-1}\left(a + \frac{k(b-a)}{n}\right) \cdot \frac{b-a}{n} = \frac{B(b-a)(a-b+n(a+b))}{2n}$$

and we note the exact value of the integral is

$$\int_a^b Bxdx = \frac{B}{2}(b^2 - a^2)$$

Subtracting the two results we find the error is

$$\frac{B(b-a)^2}{2n}$$

after some algebra.

□

18c. Richardson's Extrapolation. The trapezoidal estimate has quadratic convergence, meaning that if a definite integral with value I is estimated by $T_f(n)$ with error ϵ, then

$$T_f(n) = I + \epsilon$$

and

$$T_f(2n) \approx I + \frac{\epsilon}{4}$$

Derive the following formula for Richardson's extrapolation on the midpoint sum

$$REM_f(2n) = \frac{4M_f(2n) - M_f(n)}{3}$$

Solution:
We note that $M_f(n) \approx I + \epsilon$ and $M_f(2n) \approx I + \epsilon/4$. Thus we find

$$4T_f(2n) \approx 4I + \epsilon$$

Subtracting the first equation from this we find

$$4T_f(2n) - T_f(n) = 3I$$

Dividing by 3 yields the desired result.

$\square$