Calculus Solutions: Chapter 6.3

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Evaluate each definite integral as the limit of a sum.

1b. $\int_{-1}^{1} x^2 dx$

Solution:
From (6.23) we note that $x_k = -1 + 2k/n$ and

$$\int_{-1}^{1} x^2 = \lim_{n\to\infty} \sum_{k=1}^{n} (-1+2k/n)^2 \frac{2}{n}$$

$$= \lim_{n\to\infty} \left[\frac{8}{n^3} \sum_{k=1}^{\infty} k^2 - \frac{8}{n^2} \sum_{k=1}^{\infty} k + \frac{2}{n} \right]$$

$$= \lim_{n\to\infty} \left[\frac{8}{n^3} \left(\frac{n(n+1)(2n+1)}{6} \right) - \frac{8}{n^2} \left(\frac{n(n+1)}{2} \right) + 2 \right]$$

$$= \frac{16}{6} - \frac{8}{2} + 2 = \frac{2}{3}$$

□

1d. $\int_{1}^{4} (x - x^2) dx$

Solution:
Proceeding in the same manner as the above, we find

$$\int_{1}^{4} (x - x^2) dx = -\frac{27}{2}$$

□

Evaluate each definite integral as an area.

2b. $\int_{-2}^{4} |x| dx$

Solution:
We find the area of the triangles under the graph of $|x|$ equals 10.

□

2d. $\int_{-2}^{2} \sqrt{4-x^2}dx$

Solution:
This is a semicircle of radius 2. Thus its area is 2π.

□

2f. $\int_{-1}^{1}(3+\sqrt{1-x^2})dx$

Solution:
Finding the area of the semicircle and rectangle making up the area below the graph of the function, we find the integral value is $6+\pi/2$.

□

Rewrite the limit as a definite integral

3b. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^3}{n^4}$

Solution:
Inverting formula (6.23) we find $f(x_k) = (k/n)^3$ and $(b-a) = 1$. Thus we let $a = 0, b = 1$, from which $x_k = k/n$. Therefore $f(x) = x^3$, and we have the integral representation

$$\sum_{k=1}^{n}\frac{k^3}{n^4} = \int_0^1 x^3 dx$$

□

3d. $\lim_{n\to\infty}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \cdots + \sin\frac{n\pi}{n}\right)\frac{\pi}{n}$

Solution:
Inverting the formula (6.23) we see $b - a = \pi$ and $f(x_k) = \sin\left(\frac{k\pi}{n}\right)$. Setting $a = 0$ and $b = \pi$, we find $x_k = k\pi/n$. Thus $f(x) = \sin x$ and we have the integral representation:

$$\sum_{k=1}^{\infty}\sin k\pi n\frac{\pi}{n} = \int_0^{\pi}\sin x dx$$

□

4. Show that $\int_n^{n+1}\lfloor x\rfloor dx = n$ for $n \in \mathbb{N}$

Solution:
Using the area interpretation of the integral, we note that area under $\lfloor x\rfloor$ is given by that of a rectangle of height n and length 1. Hence, the value of the integral is n.

□

6. Show that $\int_a^b da = b - a$.

Solution:
Using the area interpretation of the integral, we note that the area under the constant function 1 on the interval $[a, b]$ is that of a rectangle of height 1 and length $b - a$. Hence the value of the integral is $b - a$.

□

9. A function f is uniformly continuous on a set S iff, for any $\epsilon > 0$ there exists $\delta > 0$ such that if $x_1, x_2 \in S$ and $|x_1 - x_2| < \delta$ then $|f(x_1) - f(x_2)| < \epsilon$. In advanced calculus it is proved that if a function f is defined and continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$. Show that each of the following functions is uniformly continuous on the given interval. That is, given $\epsilon > 0$, find $\delta > 0$ satisfying the definition of uniform continuity.

9a. $f(x) = \sin x$, $[0, 2\pi]$

Solution:
Fix $\epsilon > 0$ and let $x_1, x_2 \in [0, 2\pi]$. Then we find

$$|f(x_1) - f(x_2)| = |\sin(x_1) - \sin(x_2)|$$
$$= |2\sin((x_1 - x_2)/2)\cos((x_1 + x_2)/2)| < 2|\sin(\delta)/2 \cdot 1| = 2\sin\delta$$

Thus if we choose $\delta < \arcsin\epsilon$
we find

$$|f(x_1) - f(x_2)| < \epsilon$$

□

9b. $f(x) = 2x + 3$, $[-1, 1]$

Solution:
Fix $\epsilon > 0$ and let $x_1, x_2 \in [-1, 1]$. Then we find

$$|f(x_1) - f(x_2)| = |2(x_1 - x_2)| = 2|x_1 - x_2| < 2\delta$$

Choosing $\delta < \epsilon/2$ we find

$$|f(x_1) - f(x_2)| < \epsilon$$

□

9d. $f(x) = 9 - x^2$, $[1, 3]$

Solution:
Fix $\epsilon > 0$ and let $x_1, x_2 \in [1, 3]$. Then we find

$$|f(x_1) - f(x_2)| = |x_2^2 - x_1^2| = |x_1 - x_2| \cdot |x_2 + x_1| < 6\delta$$

Choosing $\delta < \epsilon/6$ we find

$$|f(x_1) - f(x_2)| < 6\delta$$

□