Calculus Solutions: Chapter 4.8

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2. Analyze the properties of the families of functions

$$f(x) = x^2 - ax$$

and

$$f(x) = ax - x^2$$

Solution:
For $f(x) = x^2 - ax$,

$$f'(x) = 2x - a = 0 \Leftrightarrow x = \frac{a}{2}$$

$f'' = 2 > 0$, so f has a global minimum at $x = \frac{a}{2}$. We also see that f is concave up everywhere. We see that f is decreasing for $x < \frac{a}{2}$ and increasing for $x > \frac{a}{2}$

For $f(x) = -x^2 + ax$,

$$f'(x) = -2x + a = 0 \Leftrightarrow x = \frac{a}{2}$$

$f'' = -2 < 0$, so f has a global maximum at $x = \frac{a}{2}$. We also see that f is concave down everywhere. We see that f is increasing for $x < \frac{a}{2}$ and decreasing for $x > \frac{a}{2}$

We note that these properties are obvious since f is a parabola in each case.

□

4. For a cubic function with two critical points, show that the inflection point is midway between the critical points.

Solution:
We have critical points

$$x = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$$

which have midpoint $-\frac{b}{3a}$ which is the desired result.

□

6. If a cubic function has no critical point, may it have an inflection point?

Solution:
Yes. By letting $x = -\frac{b}{3a}$ we get the desired result.

□

9. Analyze the properties of the families of functions

$$f(x) = x^3 - ax$$

and

$$f(x) = ax - x^3$$

Solution:
We use our knowledge on cubic functions. For $f(x) = x^3 - ax$, when $a \geq 0$ we get critical points at $\pm\frac{\sqrt{3a}}{3}$. Otherwise we get no critical point. The point corresponding to $\frac{\sqrt{3a}}{3}$ is a minimum and the point corresponding to $-\frac{\sqrt{3a}}{3}$ is a maximum. The inflection point is at (0,0).

For the other function the extrema are reversed.

□

12. A certain discount store specializing in fad items models sales of any particular item as a surge function. Sales of a certain clothing item rose rapidly during its first week of availability to 55 sales per day before sales began to decline. When can the store expect the sales of this item to drop below 20 per day?

Solution:
Letting $f(x) = axe^{-bx}$ and using the critical point we get $\frac{1}{b} = 1 \Rightarrow b = 1$ and $f(\frac{1}{b}) = 55 \Rightarrow a = 55e$. So we must solve the equation

$$\frac{20}{55e} = xe^{-x}$$

which yields $x = 3.16$

□

16. Verify that the asymptotes of the logistic function are as given in the text.

Solution:

$$\lim_{x \to \infty} \frac{a}{1 + be^{-kx}} = a$$

and

$$\lim_{x \to -\infty} \frac{a}{1 + be^{-kx}} = 0$$

□

22. Analyze the functions of the family

$$f(x) = \frac{a}{x+b}$$

Solution: Assume $a \neq 0$.

First we note that the function has a vertical asymptote at $x = -b$ and a horizontal asymptote of $y = 0$.

$$f'(x) = -\frac{a}{(x+b)^2} \neq 0$$

We see that f is increasing if $a < 0$ and decreasing if $a > 0$.

$$f'' = \frac{2a}{(x+b)^3} \neq 0$$

We see that f is concave down if $a < 0$ and concave up if $a > 0$.

If $a = 0$ we have the zero function.

□