Calculus Solutions: Chapter 4.5

Aaron Peterson, Stephen Taylor

November 10, 2006

2. Repeat example 72 for a square sheet of cardboard s inches on a side.

Solution:
We have

$$V = (s - 2x)^2 x = xs^2 - 4x^2 s + 4x^3$$

So

$$V' = s^2 - 8sx + 12x^2 = 0 \Leftrightarrow x = \frac{s}{2}, \frac{s}{6}$$

So $\frac{s}{6}$ is the optimal length to be cut.

□

4. A dog owner wishes to build four kennels by making a long rectangular pen and putting three fences across it (see Figure 4.31). If there is 100 feet of fencing available, what dimensions should the long rectangle be in order to enclose the maximum area?

Solution:
$A = xy$ and $2x + 5y = 100 \Leftarrow x = (100 - 5y)/2$. So $A = \frac{y(100-5y)}{2}$

$$A' = 50 - 5y = 0 \Leftrightarrow y = 10, x = 25$$

□

5. A piece of commercial property is to be enclosed by fencing on the front and two sides. Fencing for the sides costs \$2.50 per foot, and fencing for the front costs \$4.00 per foot. What are the dimensions of the largest rectangular lot that can be enclosed for \$800.

Solution:
$A = xy$ where x denotes the length of the front, and y denotes the length of the sides. We also have $4x + 5y = 800$. So $x = (800 - 5y)/4$ and $A = 200y - \frac{5}{4}y^2$

$$A' = 200 - \frac{5}{2}y = 0 \Leftrightarrow y = 80, x = 100$$

□

6. The strength of a wooden beam is proportional to the product of its width W and the square of its depth D. What are the cross-section dimensions of the strongest beam that can be cut from a cylindrical log of radius r?

Solution:
$S = kWD^2$ and $\frac{W}{4}^2 + \frac{D}{4}^2 = r^2$. $D = 2\sqrt{r^2 - \frac{W}{4}^2}$ and $A = 4kWr^2 - kW^3$.

$$A' = 4kr^2 - 3kW^2 = 0 \Leftarrow W = \sqrt{\frac{4}{3}r^2}$$

□

10. A rectangle has its base on the x-axis and its upper vertices on the curve $y = 4 - x^2$. Find the largest possible area of such a rectangle.

Solution:
$A = xy = 4x - x^3$

$$A' = 4 - 3x^2 = 0 \Leftrightarrow x = \pm\sqrt{\frac{4}{3}}$$

So the vertices are $(\sqrt{\frac{4}{3}}, \frac{8}{3})$ and $(-\sqrt{\frac{4}{3}}, \frac{8}{3})$. And the area is $A = 4\sqrt{\frac{4}{3}} - \sqrt{\frac{4}{3}}^3$

□

12. A trapezoid has three sides of length L. What is the maximum area that the trapezoid can have?

Solution:
$A = \frac{1}{2}h(b_1 + b_2)$. We note that $b_1 = L$ and the height is given by the equation

$$\left(\frac{b_2 - L}{2}\right)^2 + h^2 = L^2$$

So $h = \sqrt{L^2 - \left(\frac{b_2 - L}{2}\right)^2}$.

$$A = \frac{1}{2}\sqrt{L^2 - \left(\frac{b_2 - L}{2}\right)^2}(b_2 + L)$$

Maximizing we see that $A(2L)$ is the largest possible area.

□

14. A church window is in the shape of a rectangle surmounted by a semicircle. The semicircle is stained glass that admits half as much light per unit area as the clear glass of the rectangle. If the perimeter of the window is 204 inches, what should be the dimensions of the rectangle in order to admit the maximum amount of light?

Solution:
$204 = \frac{\pi x}{2} + x + 2y$ and

$$L = \frac{k}{2}\left(\frac{\pi}{2}\left(\frac{x}{2}\right)^2\right) + kxy = \frac{k}{2}\left(\frac{\pi}{2}\left(\frac{x}{2}\right)^2\right) + kx\left(\frac{204 - \frac{\pi x}{2} - x}{2}\right)$$

where k denotes light per unit area allowed by the rectangle.

$$L = \left(\frac{-3k\pi}{16} - \frac{k}{2}\right)x^2 + 102kx \Rightarrow L' = \left(\frac{-3k\pi}{8} - k\right)x + 102k = 0 \Rightarrow x = 46.83, y = 41.8$$

17. A box with a square base is to hold 500 cubic inches. Material for the top and bottom of the box costs 72 cents per square foot and material for the sides costs 48 cents per square foot. Find the dimensions and cost of the most economical box.

Solution:
First we convert the units to square inches. Material for the top and bottom costs .5 cents per square inch, and material for the sides costs $\frac{1}{3}$ cents per square inch.

$$C = 2b^2(.5) + \frac{1}{3}4a^2 \text{ and } 500 = ab^2$$
$$C = \frac{500}{a} + \frac{1}{3}4a^2 \Rightarrow C' = -\frac{500}{a^2} + \frac{8}{3}a = 0$$
$$\Rightarrow \frac{8}{3}a^3 - 500 = 0 \Rightarrow a = (\frac{3}{8}500)^{(1/3)}, b = (\frac{8}{3})^{1/6}500^{(1/3)}$$

where b denotes the side length of the bottom of the box.

□

18. A tin can is a cylinder with two ends. Show that the can with the least amount of material and containing a given volume has height equal to the diameter.

Solution:

$$V = \pi r^2 h \text{ and } M = 2\pi r^2 + 2r\pi h$$
$$h = \frac{V}{\pi r^2} \Rightarrow M = 2\pi r^2 + 2r\pi\frac{V}{\pi r^2} \Rightarrow V = 2\pi r^3$$

By making the appropriate substitution we see that $2r = h$.

□

20. In a particular apartment complex of 80 units, it is found that all units remain occupied when the rent is $400 per month. For each $20 increase in the rent, one unit becomes vacant, on the average. Occupied units require $40 per month for maintenance, while vacant units require none. Fixed costs for the complex are \$24,000 per month. What rent should be charged for maximum profit?

Solution:
Quantity rented q as a function of price p is given by

$$q = -\frac{1}{20}p + 80$$

So profits are given by

$$P = R - C = -\frac{1}{20}p^2 + 80p + 2p - 27200$$

Optimizing we see that the price should be \$820 per month.

□

24. A tapestry 14 feet high hangs on a wall with its lower edge 2 feet above eye level. How far from the wall should a viewer stand so as to get the best view of the tapestry?

Solution:
This problem is similar to Example 78. So we will use a figure like figure 4.29. Our equation is $\alpha = \tan^{-1}\frac{16}{x} - \tan^{-1}\frac{2}{x}$

$$\alpha' = \frac{-\frac{16}{x^2}}{1+\frac{256}{x^2}} + \frac{\frac{2}{x^2}}{1+\frac{4}{x^2}} = \frac{\frac{448}{x^4} - \frac{14}{x^2}}{(1+\frac{256}{x^2})(1+\frac{4}{x^2})} \Rightarrow x = \sqrt{32} = 4\sqrt{2}$$

□

26. Find a point on the curve $y = \sqrt{4-x}$ that is nearest the origin.

Solution:

$$d = \sqrt{x^2+4-x} \Rightarrow d' = \frac{1}{2}(x^2 - x + 4)(2x-1) = 0 \Rightarrow x = \frac{1}{2}, y = \sqrt{\frac{7}{2}}$$

□

28. A crook decides to hold up a pedestrian along a certain block. At each end of the block there is a street light, but there are no other lights on that block. Maintenance problems being as they are, one of the street lights is six times as bright as the other. The intensity of light at a point is jointly proportional to the brightness of the source and the reciprocal of the square of the distance from the source, and the intensities from different sources are additive. if the crook wants to accost his victim at the darkest point along the street, where should he station himself?

Solution:

The intensity as a function of x, where x is the distance from the brighter light, is given by

$$I(x) = \frac{6k}{x^2} + \frac{k}{(L-x)^2}$$

where L is the length of the block.

$$I' = \frac{2kx^3 - 12k(L-x)^3}{x^3(L-x)^3} = 0 \Rightarrow x = \frac{6^{(1/3)}L}{1+6^{(1/3)}}$$

□

30. A student estimates that her learning efficiency E depends on the number t of hours since she awoke by the formula

$$E(t) = 62.7t - 16t^2 + 1.47t^3 - 0.044t^4$$

If she awakens at 6:00 am, when does she reach her maximum learning efficiency?

Solution:

$$E'(t) = 62.7 - 32t + 4.41t^2 - 1.76t^3 = 0 \Rightarrow x = 2.063$$

She reaches her maximum learning efficiency shortly after 8:00 am.

□

31. A fence 6 feet high is 2 feet from a building. How long is the shortest ladder that will rest on the ground outside the fence and against the building, clearing the fence?

Solution:
If x denotes the distance between the base of the ladder and the fence, and h denotes the height at which the ladder meets the wall, we have l representing the length of the ladder given by

$$l(x) = \sqrt{\frac{36(x+2)^2}{x^2} + (x+2)^2}$$

and

$$l' = \frac{2(2+x) + \frac{72(2+x)}{x^2} - \frac{72(2+x)^2}{x^3}}{2\sqrt{(2+x)^2 + \frac{36(2+x)^2}{x^2}}}$$

We see that $x = 4.16017$ feet. So $l = 10.8$.

□

32. Saturn is roughly a sphere of radius 56,500 km. The rings of Saturn are circular and lie in the plane of Saturn's equator. The innermost edge of the rings has a radius of 88,500 km, and the outermost edge of the rings has radius 138,800 km.

a) From where on Saturn's surface (assuming it had a surface) could an observer see the inner edge of the rings?

Solution:
We consider a cross sectional circle of the planet. Construct the tangent line from point representing the inner ring $(88500, 0)$ to the circle. This line hits the circle at a point (x, y). The distance from the point to the circle is given by

$$(x - 88500)^2 + y^2 = 88500^2 - 56500^2$$

We use the fact that $x^2 + y^2 = 56500^2$ to determine that
$(x, y) = (36070.6, 43487.5)$

Now we use symmetry and we find the angle from $(0, 0)$ to (x, y). We get 43 degrees north and south latitude.

□

b) From where on Saturn's surface would the rings be observable at all?

Solution:
Repeat this process using the point $(138, 800, 0)$ instead.

□

c) From where on Saturn's surface would the rings of Saturn appear to have the greatest width? Could an observer with this view see the inner edge?

Solution:
We want to choose the point (x, y) on the circle where the angle at (x, y) in the triangle formed by $(x, y), (88500, 0),$ and $(138800, 0)$ is maximized. The function to maximize is

$$\arccos\left(\frac{138800^2 - 2(138800)(88500) + 88500^2 - 138800^2 - 88500^2 - 2(56500)^2 + 2(88500)x + 2(138800)x}{-2\sqrt{(56500^2 - 2(88500)x + 88500^2)(56500^2 - 2(138800)x + 138800^2)}}\right)$$

This yields that you should stand at 33.9 degrees north or south latitude. So the inner rings are visible from there.

□

34. Two ranches are located on the same side of a straight river, one 8 miles from the river and the other 2 miles from the river. The ranches are 10 miles apart, as the crow flies. It is desired to put a pumping station on the river and run pipelines to the ranches.

a) If each ranch is to have its own pipeline from the pumping station, where on the river should the pumping station be placed so as to minimize the total length of the pipelines?

Solution:
The function to optimize is

$$\sqrt{x^2 + 64} + \sqrt{(8 - x)^2 + 4}$$

where x is the distance along the river from the pumping station to the point on the river closest to the first ranch. We see that x should be $\frac{32}{5}$.

□

b) Solve the problem if it is possible to run a single pipeline from the pumping station to one ranch and then directly to the other ranch.

Solution:
We put the pumping station on the river at the closest point to the second ranch or $x = 8$.

□

c) Solve the original problem without calculus by imagining the ranches to be on opposite sides of the river.

Solution:
The problem only makes sense when the river has width 0 in which case the answer is obvious.

□

35. Figure 4.34 shows the function $f(v)$ representing the energy expended by a bird in level flight through calm air, measured in joules per second, at velocity v, measured in meters per second.

a) Explain why the graph may be shaped as it is, in terms of how a bird flies.

Solution:
Hovering takes more energy than flight because of lift.

□

b) Let v_0 be the velocity that minimizes $f(v)$. Tell approximately where v_0 is located on the graph.

Solution:
Near the first tic mark

□

c) Let $a(v)$ be the energy expended by the same bird, but measured in joules per meter. What is the relationship between $a(v)$ and $f(v)$?

Solution:

$$a(v) = \frac{f(v)t}{vt} = \frac{f(v)}{v}$$

□

e) Let v_1 be the velocity that minimizes $a(v)$. Tell where v_1 is located.

Solution:
Near the second tic mark

□

f) Under what circumstances should the bird try to minimize $f(v)$ or $a(v)$?

Solution:
Minimize f if you would like to stay aloft, and minimize a if you would like to get somewhere.

□