Calculus Solutions: Chapter 4.4

Aaron Peterson, Stephen Taylor

October 27, 2006

1. For each function, locate the extrema and find the extreme values, and classify each extremum as local or global.

b) $f(x) = x^2 + 2x + 3$

Solution:
$f'(x) = 2x + 2$, so f has a critical point at $x = -1$.
$f''(x) = 2 > 0$, so $f(-1) = 1 - 2 + 3 = 2$ is a global minimum.

□

d) $f(x) = 4x^3 - 21x^2 - 24x + 5$

Solution:
$f'(x) = 12x^2 - 42x - 24$, so f has critical points at $x = 4, -\frac{1}{2}$.
$f''(x) = 24x - 42$. $f''(4) > 0$ and $f''(-\frac{1}{2}) < 0$. f has a local min at $f(4) = -171$ and a local max at $f(-\frac{1}{2}) = 11.25$.

□

f) $f(x) = x^3 - x^6$

Solution:
$f'(x) = 3x^2 - 6x^5$, so f has critical points at $x = 0, \frac{1}{2}^{\frac{1}{3}}$.
$f''(x) = 6x - 30x^4$.

$$f''(\frac{1}{2}^{\frac{1}{3}}) = 6\frac{1}{2}^{\frac{1}{3}} - 30\frac{1}{2}^{\frac{4}{3}} < 0$$

Since $f''(0) = 0$ we must apply the first derivative test. Clearly we can find an open interval containing 0 so that $f'(x) \geq 0$ on the interval. So f does not attain a min or a max at 0. f attains a global max at $f(\frac{1}{2}^{\frac{1}{3}})$.

□

h) $f(x) = x^2 - 5x + 6$

Solution:
$f'(x) = 2x - 5$, so f has a critical point at $x = \frac{5}{2}$.
$f''(x) = 2$. So f has a global min of $f(\frac{5}{2}) = -\frac{1}{4}$.

□

j) $f(x) = x - \frac{1}{x}, x \in (0, 3]$

Solution:
$f'(x) = 1 + \frac{1}{x^2}$. f does not have any critical points. We note that f attains a global max of $f(3) = \frac{8}{3}$ since f' is always positive on the domain.

□

l) $f(x) = (x+2)^{1/5}, x \in (-4, \infty)$

Solution:
$f'(x) = \frac{1}{5}(x+2)^{-4/5}$, so f has a critical point at $x = -2$. We note that f' is always positive, so we do not attain a min or a max at $x = -2$

□

2. For each function, locate the extrema and find the extreme values, and classify each extremum as local or global.
b) $f(x) = x + \sqrt{1 - x^2}$

Solution:
$f'(x) = 1 - \frac{x}{\sqrt{1-x^2}}$, so f has critical points at $x = \pm 1, \sqrt{\frac{1}{2}}$ $f''(x) = \frac{\frac{1-2x^2}{\sqrt{1-x^2}}}{1-x^2}$. $f''(\sqrt{\frac{1}{2}}) = 0$, so the test fails. We note that $x = -1$ is clearly a global min. Also note that $f'(x) > 0$ for $x < \sqrt{\frac{1}{2}}$ and $f'(x) < 0$ for $x > \sqrt{\frac{1}{2}}$. f attains a global max at $\sqrt{\frac{1}{2}}$.

□

d) $f(x) = \sin x - \cos 2x$

Solution:

$$f'(x) = \cos x + 2\sin 2x = 0 \Leftrightarrow x = \pm\frac{\pi}{2}, -\sin^{-1}\frac{1}{4}$$
$$f''(x) = -\sin x + 4\cos 2x \Rightarrow f''(-\sin^{-1}\frac{1}{4}) > 0$$

So f has a global min of $f(-\sin^{-1}\frac{1}{4}) = -1.125$. Also $f''(\pm\frac{\pi}{2}) < 0$, so f has local maximums $f(-\frac{\pi}{2}) = 0$ and global maximums $f(\frac{\pi}{2}) = 2$.

□

f) $f(x) = x - 2\sin x$

Solution:

$$f'(x) = 1 - 2\cos x = 0 \Leftrightarrow x = \pm\frac{\pi}{3}$$
$$f''(x) = 2\sin x \Rightarrow 2\sin(-\frac{\pi}{3}) < 0 \text{ and } 2\sin(\frac{\pi}{3}) > 0$$

So f has global mins at $f(\frac{\pi}{3} + 2n\pi)$ and global maximums at $f(-\frac{\pi}{3} + 2n\pi)$.

□

h) $f(x) = \frac{1}{1+\sec^2 x}$

Solution:

$$f'(x) = -\frac{2\sec^2 x\tan x}{(1+\sec^2 x)^2} = 0 \Leftrightarrow \sin x = 0 \Leftrightarrow x = n\pi$$

$$f''(x) = -\frac{2(2\sec^2 x\tan^2 x + \sec^4 x)(1+\sec^2 x)^2 - (2\sec^2 x\tan x)(4\sec^2 x\tan x)(1+\sec^2 x)}{(1+\sec^2 x)^4}$$

$f''(n\pi) < 0$ so f attains local maximums of $f(n\pi) = \frac{1}{2}$.

□

j) $f(x) = -2xe^{-3x}$

Solution:

$$f'(x) = -2e^{-3x} + 6xe^{-3x} = 0 \Leftrightarrow x = \frac{1}{3}$$
$$f''(x) = 12e^{-3x} - 18xe^{-3x} > 0 \text{ at } x = \frac{1}{3}$$

So f attains a global min of $-\frac{2}{3}e^{-1}$.

□

l) $f(x) = 3e^{x^2/4}$

Solution:

$$f'(x) = -\frac{3}{2}xe^{-x^2/4} = 0 \Leftrightarrow x = 0$$
$$f''(x) = -\frac{3}{2}e^{-x^2/4} + \frac{3}{4}x^2e^{-x^2/4} < 0 \text{ for } x = 0$$

So f attains a global max of 3.

□

6. If a function is continuous on a closed interval and has exactly one local maximum there, must the maximum be global?

Solution:
Yes. If the maximum is not global, then it is not the only maximum.

□

8. The Minimax Game Player P1 chooses a real number x, and player P2 chooses a real number y. A score S is computed using the formula

$$S = x^2 - xy - 2y^2 + 9x + 27y - 90$$

If the score is positive, P1 pays P2 the value of the score; if the score is negative, P2 pays P1 the absolute value of the score. There are various possible rules governing the play:
Rule 1. P1 announces his choice and then P2 chooses.
Rule 2. P2 announces his choice and then P1 chooses.
Rule 3. Both players write down their choices before either sees the other's choice.

a) What is the best strategy for P2 if Rule 1 is used?

Solution:
For a given x, $S'(y) = -x - 4y + 27$, so to maximize, P2 should choose $y = \frac{27-x}{4}$.

□

b) What is the best strategy for P1 if Rule 1 is used?

Solution:
P1 should choose $x = -1$

□

c) What is the best strategy for P1 is Rule 2 is used?

Solution:
Similarly P1 should choose $\frac{y-9}{2}$

□

d) What is the best strategy for P2 if Rule 2 is used?

Solution:
P2 should choose 7.

□

e) What is the best strategy for each player if Rule 3 is used?

Solution:
By parts b and d, P1 should choose -1, and P2 should choose 7.

□