Calculus Solutions: Chapter 4.2

Aaron Peterson, Stephen Taylor

October 27, 2006

1. Find the critical points of the following functions.
b) $f(x) = 4 - x^2$

Solution:
$f'(x) = -2x$ So $f(x)$ has a critical point at $x = 0$.

□

d) $f(x) = x^2 - \frac{1}{x}$

Solution:
$f'(x) = 2x + \frac{1}{x^2}$ So we must solve the equation

$$2x = -\frac{1}{x^2}$$

We determine that $f(x)$ has a critical point of $x = -\frac{1}{\sqrt[3]{2}}$ and $x = 0$ since the derivative is undefined there.

□

f) $g(x) = \sqrt{x^2 + 1}$

Solution:
$g'(x) = \frac{x}{\sqrt{x^2+1}}$ So $g(x)$ has a critical point of $x = 0$ since the derivative is 0 there.

□

h) $g(x) = (x + 2)^{\frac{1}{5}}$

Solution:
$g'(x) = \frac{1}{5}(x + 2)^{-\frac{4}{5}}$ which is never 0, but is undefined when $x = -2$. So $g(x)$ has a critical point of $x = -2$.

□

j) $f(x) = 2\cos x + \sin 2x$

Solution:
$f'(x) = -2\sin x + 2\cos 2x$ We must solve

$$2\sin x = 2\cos 2x \Leftrightarrow x = \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi, \frac{3\pi}{2} + 2n\pi$$

□

l) $f(x) = x + 2\sin x$

Solution:
$f'(x) = 1 + 2\cos x$ We must determine where $\cos x = -\frac{1}{2}$ So $f(x)$ has critical points at $x = \frac{2\pi}{3} + 2n\pi$.

□

n) $g(x) = 3e^{-x}$

Solution:
$f'(x) = -3e^{-x}$ which is defined everywhere and never 0, so $f(x)$ has no critical points.

□

p) $g(x) = e^{\sin x}$

Solution:
$g'(x) = (\cos x)e^{\sin x}$ which is defined everywhere and 0 when $\cos x = 0$. $g(x)$ has critical points at $x = (2n+1)\frac{\pi}{2}$.

□

3. Use Rolle's Theorem to prove that the function $f(x) = 4 - x - 6x^3$ has only one real zero.

Solution:
Suppose a and b are real zeros of f. Then there exists at least one real number $c \in (a, b)$ such that $f'(c) = -1 - 18x^2 = 0$. This does not occur, so f can have at most one real zero. Since the order of f is odd, f has at least one real zero. So we see that f has exactly one real zero.

□

4b. Explain why Rolle's Theorem is not satisfied by $f(x) = \frac{\sin x}{x}$ for $0 < x \leq \pi$ and $f(x) = 0$ for $x = 0$ on the interval $[0, \pi]$.

Solution:

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{\sin x}{x} = 1 \neq 0$$

So $f(x)$ is not continuous on $[0, \pi]$.

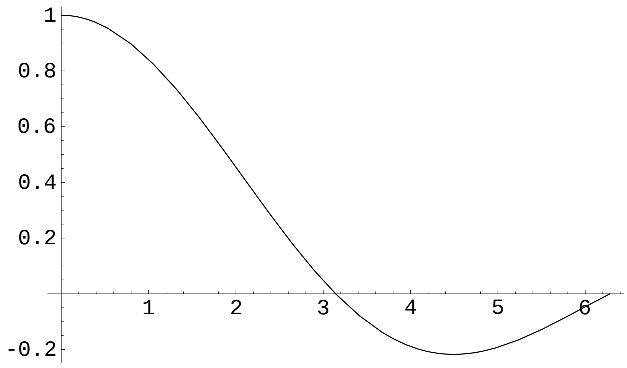

Figure 1: $\sin x$ for $x \in [0, 2\pi]$

□

6. Two cars are traveling on the freeway in the same direction. In the course of the trip, each car passes the other twice. Prove that at some point in time, the cars have the same acceleration.

Solution:
Let $f(t), g(t)$ represent the position of the first and second car respectively as functions of time. Then $f'(x), g'(x)$ represent the velocity of the first and second car respectively. Since the cars pass each other twice, we know by problem 5 that there are two points in time a, b at which the cars have the same velocity. Define $h(x) \equiv g'(x) - f'(x)$. So $h(a), h(b) = 0$. Applying Rolle's Theorem, there exists a point in time c such that

$$h'(c) = 0 \Rightarrow f''(c) = g''(c)$$

So there exists a point in time at which the two cars have the same acceleration.

□

7. Find some c in the given interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

b) $f(x) = \cos x, (0, 2)$

Solution:
$f'(x) = -\sin x$ and

$$\frac{f(b) - f(a)}{b - a} = \frac{\cos 2 - 1}{2} = -0.708073$$

So letting $c = 0.786766$ we have

$$-\sin(0.786766) = -.708073$$

□

d) $f(x) = \ln(1 + x), (0, e)$

Solution:
$f'(x) = \frac{1}{1+x}$ and

$$\frac{f(b) - f(a)}{b - a} = \frac{ln(1 + e)}{e}$$
$$\frac{1}{c + 1} = \frac{ln(e + 1)}{e}$$
$$c + 1 = \frac{e}{ln(e + 1)}$$
$$c = \frac{e}{ln(e + 1)} - 1$$

□

8. Explain why no point c exists in the given interval $[a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

b) $f(x) = \sigma(x), [-1, 1]$

Solution:

$$\frac{f(b) - f(a)}{b - a} = \frac{1 - (-1)}{1 - (-1)} = 1$$

but $f'(x) = 0$ where it is defined.

□

d) $f(x) = x^{\frac{2}{3}}, [-1, 1]$

Solution:

$$\frac{f(b) - f(a)}{b - a} = \frac{1 - (-1)}{1 - (-1)} = 1$$

but $f'(x) = \frac{2}{3}x^{-\frac{1}{3}} < 1$

□

10. Prove that if function f is differentiable on $[a, b]$ and there is some $M > 0$ such that $|f'(x)| < M$ for all $x \in (a, b)$, then for any $x \in (a, b)$,

$$|f(x) - f(a)| < M|x - a|$$

Solution:
By 4.40

$$f(x) - f(a) = f'(c)(x - a)$$

So

$$|f(x) - f(a)| = |f'(c)(x - a)| < M|x - a|$$

□