Calculus Solutions: Chapter 3.2

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1.Differentiate the following functions

a) $f(x) = x^5$

Solution: Applying the power rule, we have

$$f'(x) = 5x^4$$

□

b) $f(x) = 2x^4$

Solution:
Applying the power rule, we have

$$f'(x) = 8x^3$$

□

c) $f(x) = x^{-7}$

Solution:
Applying the power rule for negative integers, we have

$$f'(x) = -7x^{-8}$$

□

d) $f(x) = -5x^{-3}$

Solution:
Applying the power rule for negative integers , we have

$$f'(x) = 15x^{-4}$$

□

e) $y = 6$

Solution:
Applying the derivative of a constant rule we have

$$y' = 0$$

□

f) $y = 8\pi$

Solution:
Applying the derivative of a constant rule we have

$$y' = 0$$

□

g) $y = 2x^2 + 3x - 6$

Solution:
Applying the power rule and linearity of the derivative operator we have

$$y' = 4x + 3$$

□

h) $y = 3x^5 - 5x - 20$

Solution:
Applying the power rule and the linearity of the derivative operator we have

$$y' = 15x^4 - 5$$

□

i) $g(x) = x^2 + x^{-2}$

Solution:
Applying both the positive and negative power rules and linearity we have

$$g'(x) = 2x - 2x^{-3}$$

□

j) $g(x) = x^3 + 5 - x^{-3}$

Solution:
Applying both the positive and negative power rules and linearity we have

$$g'(x) = 3x^2 + 3x^{-4}$$

□

k) $g(x) = 2x(x-3)$

Solution:
Applying the power rule and linearity we have

$$g(x) = 2x(x-3) = 2x^2 - 6x$$
$$g'(x) = 4x - 6$$

□

l) $g(x) = x^3(4x+3)$
Applying the power rule and linearity we have

$$g(x) = x^3(4x+3) = 4x^4 + 3x^3$$
$$g'(x) = 16x^3 + 9x^2$$

□

2. Differentiate the following functions

Solution:
a) $f(x) = (x^2 + 3x + 1)(x^8 + x^{-8})$

Solution:
Expanding $f(x)$ and applying both the positive and negative power rules, we find

$$f(x) = x^{10} + 3x^9 + x^8 + x^{-6} + 3x^{-7} + x^{-8}$$
$$f'(x) = 10x^9 + 27x^8 + 8x^7 - 6x^{-7} - 21x^{-8} - 8x^{-9}$$

□

b) $f(x) = (2x^5 - 4x^2)(3x^{10} + x^2)$

Solution:
Expanding $f(x)$ and applying the power rule we find

$$f(x) = 6x^{15} - 12x^{12} + 2x^7 - 4x^4$$
$$f'(x) = 90x^{14} - 144x^{11} + 14x^6 - 16x^3$$

□

c) $f(x) = (2x+1)^2$

Solution:
Expanding $f(x)$ and applying the power rule we find

$$f(x) = 4x^2 + 4x + 1$$
$$f'(x) = 8x + 4$$

□

d) $f(x) = (2x + 1)^3$

Solution:
Expanding $f(x)$ and applying the power rule we find

$$f(x) = 1 + 6x + 12x^2 + 8x^3$$
$$f'(x) = 6 + 24x + 24x^2$$

□

e) $y = \frac{1}{2x-1}$

Solution:
Applying the quotient rule we find

$$y' = \frac{(2x-1)\frac{d}{dx}(1) - \frac{d}{dx}(2x-1)}{(2x-1)^2} = -\frac{2}{(2x-1)^2}$$

□

f) $y = (x^4 + x^2 - 1)^{-1}$

Solution:
Applying the quotient rule we find

$$y' = -\frac{4x^3 + 2x}{(x^4 + x^2 - 1)^2}$$

□

g) $y = \frac{x}{x+1}$

Solution:
Applying the quotient rule we find

$$y' = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2}$$

□

h) $y = \frac{x^2-1}{2x-3}$

Solution:
Applying the quotient rule we find

$$y' = \frac{2x^2 - 6x + 2}{(2x-3)^2}$$

□

i) $g(x) = \frac{2x}{x^2-1}$

Solution:
Applying the quotient rule we find

$$y' = -\frac{2(x^2+1)}{(x^2-1)^2}$$

$\square$

j) $g(x) = \frac{2x^3-x-1}{x^2+4x}$

Solution:
Applying the quotient rule we find

$$g'(x) = \frac{2x^4+16x^3+x^2+2x+4}{x^2(4+x)^2}$$

$\square$

k) $g(x) = \frac{ax}{bx+c}$

Solution:
Applying the quotient rule we find

$$g'(x) = \frac{ac}{(c+bx)^2}$$

$\square$

l) $g(x) = \frac{x}{x^2+a^2}$

Solution:
Applying the quotient rule we find

$$g'(x) = \frac{a^2-x^2}{(a^2+x^2)^2}$$

$\square$

m) $F(x) = x(x+1)(x^2+1)$

Solution:
Expanding and applying the power rule we have

$$F(x) = x^4+x^3+x^2+x$$
$$F'(x) = 4x^3+3x^2+2x+1$$

$\square$

n) $F(x) = (x-a)(x-b)(x-c)$

Solution:
Expanding and applying the power rule we have

$$F(x) = x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc$$
$$F'(x) = 3x^2 - 2(a+b+c)x + (ab+ac+bc)$$

□

4. Find the second derivative

a) $y = 2x^2 + 3x - 6$

Solution:
Applying the power rule we find

$$y' = 4x + 3$$
$$y'' = 4$$

□

b) $y = 3x^5 - 5x - 20$

Solution:
Applying the power rule we find

$$y' = 15x^4 - 5$$
$$y'' = 60x^3$$

□

c) $f(x) = 2x^4$

Solution:
Applying the power rule we find

$$f'(x) = 8x^3$$
$$f''(x) = 24x^2$$

□

d) $f(x) = x^{-7}$

Solution:
Applying the power rule for negative exponents we have

$$f'(x) = -7x^{-8}$$
$$f''(x) = 56x^{-9}$$

□

e) $g(x) = \frac{2x}{x^2-1}$

Solution:
Applying the quotient rule we find

$$g'(x) = -\frac{2(x^2+1)}{(x^2-1)^2}$$
$$g''(x) = \frac{4x^3+12x}{(x^2-1)^3}$$

□

f) $g(x) = \frac{ax}{bx+c}$

Solution:
Applying the quotient rule we find

$$g'(x) = \frac{ac}{(bx+c)^2}$$
$$g''(x) = -\frac{2abc}{(bx+c)^3}$$

Find a formula for $f^{(n)}(x)$

5a. $f(x) = \frac{1}{x+a}$

Solution:
By the quotient rule we find

$$f'(x) = \frac{-1}{(x+a)^2}$$
$$f''(x) = \frac{2}{(x+a)^3}$$
$$f'''(x) = -\frac{6}{(x+a)^4}$$
$$f^{(4)}(x) = \frac{24}{(x+a)^5}$$

Thus we identify the patten to be

$$f^{(n)}(x) = (-1)^n \frac{n!}{(x+a)^{n+1}}$$

□

5d. $f(x) = x^k$

Solution:
By the power rule we compute

$$f'(x) = kx^{k-1}$$
$$f''(x) = k(k-1)x^{k-2}$$
$$f'''(x) = k(k-1)(k-2)x^{k-3}$$
$$f^{(4)}(x) = k(k-1)(k-2)(k-3)x^{k-4}$$

Thus we identify the pattern to be

$$f^{(n)}(x) = \frac{k!}{(k-n)!}x^{k-n}$$

□

Thus we note for $i < k$

$$f^{(i)}(x) = \frac{k!}{(k-i)!}x^{k-i}$$

For $i = k$ we find

$$f^{(k)}(x) = k!$$

For $i > k$ we have

$$f^{(i)}(x) = 0$$

6. Find an equation of the line tangent to the given curve at the given point

We note that the slope of the tangent line to a curve at a point is equal to its derivative there.

a) $y = x^3 - x, (1, 0)$

Solution:
Calculating $f'(x) = 3x^2 - 1$, we find $f'(1) = 2$. Thus the tangent line has the form

$$y = 2x + b$$

since $(1, 0)$ is a point on the line

$$0 = 2 + b \to b = -2$$

Thus the desired line is given by the equation

$$y = 2x - 2$$

□

b) $y = x^{-2}, (-1, 1)$

Solution:
Calculating $f'(x) = -2x^{-3}$, we find $f'(-1) = 2$. Thus the tangent line has the form

$$y = 2x + b$$

since $(-1, 1)$ is a point on the line

$$1 = -2 + b \rightarrow b = 3$$

Thus the desired line is given by the equation

$$y = 2x + 3$$

c) $f(x) = 3x^4 - \frac{1}{x}$, $(1, 2)$

Solution:
Calculating $f'(x) = 12x^3 + x^{-2}$, we find $f'(1) = 13$. Thus the tangent line has the form

$$y = 13x + b$$

since $(1, 2)$ is a point on the line

$$2 = 13 + b \rightarrow b = -11$$

Thus the desired line is given by the equation

$$y = 13x - 11$$

□

d) $f(x) = \frac{x^2}{x-1}, (2, 4)$

Solution:
Calculating $f'(x) = \frac{x^2-2x}{(x-1)^2}$, we find $f'(2) = 0$. Thus the tangent line has the form

$$y = b$$

since $(2, 4)$ is a point on the line we note $b = 4$.
Thus the desired line is given by the equation

$$y = 4$$

□

Find the rate of change

8b. of the volume of a sphere with respect to its radius.

Solution:
We note the formula for the volume of a sphere in $\mathbb{R}^3$ is given by

$$V(r) = \frac{4}{3}\pi r^3$$

Thus the rate of change of volume with respect to the radius of the sphere is

$$\frac{d}{dr}V(r) = 4\pi r^2$$

which is also the equation for the surface area of a sphere.

□

8c. of the volume of a cube with respect to its edge.

Solution:
We note the formula for the volume of a cube in $\mathbb{R}^3$ is given by

$$V(e) = e^3$$

where e is the edge length of the cube. Thus the rate of change of volume with respect to edge length is given by

$$\frac{d}{de}V(e) = 3e^2$$

Note that this is half the surface area of the cube.

□

17. Derive formula (3.14) for the derivative of the product of three functions.

Solution:
Formula (3.14) states for three functions $f(x), g(x), h(x)$

$$(fgh)' = f'gh + fg'h + fgh'$$

Applying the product rule to f and gh we find

$$(f[gh])' = f(gh)' + ghf' = f(gh' + hg') + ghf' = f'gh + fg'h + fgh'$$

which is the desired result.

□

18. Derive formula (3.15) for the derivative of the product of four functions.

Solution:
Formula (3.15) states

$$(uvwz)' = u'vwz + uv'wz + uvw'z + uvwz'$$

We apply the product rule to the functions u and vwz to find

$$\begin{aligned}(u[vwz]') &= u(vwz)' + vwzu' = u(v'wz + vw'z + vwz') + vwzu' \\ &= u'vwz + uv'wz + uvw'z + uvwz'\end{aligned}$$

which is the desired result.

□

19. Can you write a formula for the derivative of the product of n functions?

Solution:
Based on the preceding two examples, we infer the derivative of the product of n functions $f_1, f_2, \ldots, f_n$ is given by

$$(f_1 f_2 \cdots f_n) = f_1' f_2 \cdots f_n + f_1 f_2' \cdots f_n + \cdots + f_1 f_2 \cdots f_n'$$

□

25. Prove that the nth derivative of a polynomial function of degree n is a constant. What is the $(n+1)$th derivative of the same polynomial?

Solution:
Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$.
Then

$$\begin{aligned}f'(x) &= a_1 + 2a_2 x + \cdots + n a_n x^{n-1} \\ f''(x) &= 2a_2 + \cdots + n(n-1) a_n x^{n-2} \\ f^{(n)}(x) &= n(n-1)(n-2)\cdots 2 \cdot 1 x^0 = n!\end{aligned}$$

Thus the n-th derivative of a polynomial of degree n is constant. The $(n+1)$th derivative is 0 since the derivative of any constant is 0.

□