Calculus Solutions: Chapter 2.7

Aaron Peterson, Stephen Taylor

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1. Determine whether the given function is continuous on the given interval. Sketch the graph.

b) $f(x) = \llcorner x \lrcorner$, [0,1]

Solution:
Note that $f(1) = 1$ but

$$\lim_{x \to 1-} f(x) = 0$$

So $f(x)$ is not continuous at 1.

□

d) $f(x) = \llcorner x \lrcorner$, $[\frac{1}{2}, \frac{3}{2}]$

Solution:
Similar to part b), $f(x)$ is not continuous at 1.

□

f) $f(x) = \frac{1}{x}$, (0,1)

Solution:
$f(x)$ is a rational function defined on (0,1), so $f(x)$ is continuous on (0,1).

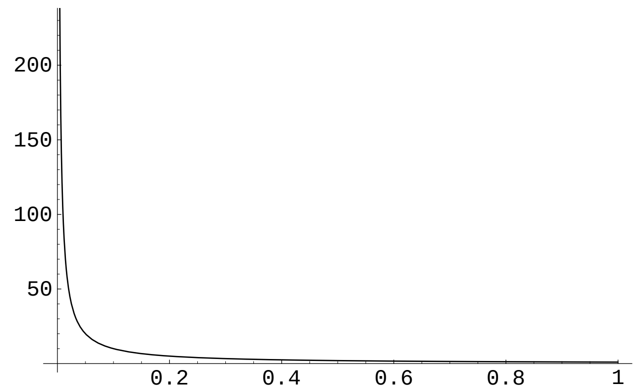

Figure 1: $\frac{1}{x}$ for $x \in (0, 1)$

$\square$

h) $f(x) = \frac{1}{x}$, $[-1, -\frac{1}{2}]$

Solution:
$f(x)$ is a rational function defined on $[-1, -\frac{1}{2}]$, so $f(x)$ is continuous on $[-1, -\frac{1}{2})$.

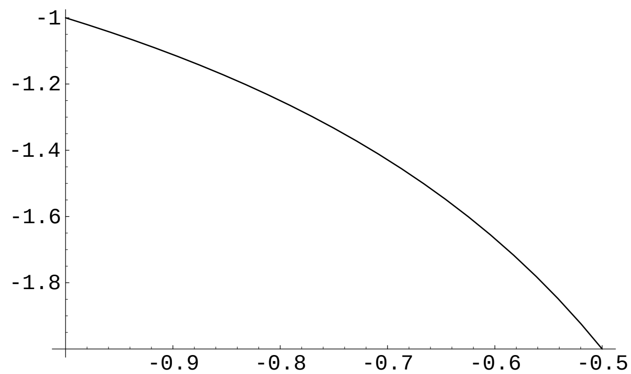

Figure 2: $\frac{1}{x}$ for $x \in [-1, -\frac{1}{2}]$

$\square$

2. Explain why each function fails to have both a maximum and a minimum on the given interval.
a) $f(x) = \frac{1}{x}, (1, 2)$

Solution:
Suppose $f(x)$ has a maximum at c given by $\frac{1}{c}$. Let $d = \frac{c+1}{2}$. $d < c$ and $d \in (1, 2)$, but $\frac{1}{d} > \frac{1}{c}$. So $f(x)$ has no maximum on (1,2). By a similar proof $f(x)$ has no minimum on (1,2).

$\square$

b) $f(x) = \sin x$ for $x \neq \frac{\pi}{2}, \frac{3\pi}{2}$ and $f(x) = 0$ for $x = \frac{\pi}{2}, \frac{3\pi}{2}$, $[0, 2\pi]$

Solution:
Suppose $f(x)$ has a maximum value of $\sin c$.
Case I: $c < \frac{\pi}{2}$

$$\sin(\frac{\frac{\pi}{2} + c}{2}) > \sin c$$

Case II: $\frac{\pi}{2} < c < \frac{3\pi}{2}$

$$\sin(\frac{c - \frac{\pi}{2}}{2}) > \sin c$$

Case III: $\frac{3\pi}{2} < c$

$$\sin c \leq 0 < \sin \frac{\pi}{4} = 1$$

Note that 0 is not a maximum value of $f(x)$ since $0 < \sin \frac{\pi}{4} = 1$.
The case for the minimum is similar.

$\square$

3b. Suppose that the functions f and g are defined and continuous in an open interval containing c.
b) If $f(c) < 0$, show that there is an open interval (a, b) containing c such that if $x \in (a, b)$ then $f(x) < 0$.

Solution:
If there is no such interval, then either

$$\lim_{x \to c+} f(x) \geq 0$$

or

$$\lim_{x \to c-} f(x) \geq 0$$

Neither of these can occur since

$$\lim_{x \to c} f(x) < 0$$

So we know that there exists such an interval.

□

5. A rocket is launched into the air and eventually burns all its fuel. Prove that at some point in the rocket's flight the amount of fuel left is equal numerically to the distance traveled at that point.

Solution:
Let $f(d)$ represent fuel remaining as a function of distance traveled. $f(d)$ is continuous on $[0, D]$ where D is the distance traveled when all the fuel is burned. Define $g(d) \equiv d$ on $[0, D]$. Define $h(d) \equiv f(d) - g(d)$ continuous on $[0, D]$. Clearly $h(0) = f(0) - g(0) > 0$ and $h(D) = f(D) - g(D) < 0$. So by the Intermediate Value Theorem there exists $c \in [0, D]$ such that $h(c) = 0$. So $f(c) = g(c) = c$. c is the desired point.

□

6b. We say that function f defined on interval I is bounded above on interval I in case there is a number B such that if $x \in I$ then $f(X) < B$. We say f is bounded below on I in case there is a number b such that if $x \in I$ then $f(x) > b$. We say f is bounded if f is bounded above and below.
b) Use the definition of continuity to prove that if f is defined and continuous in an open interval containing c, then for every $\epsilon > 0$ there exists a $\delta > 0$ such that f is bounded on the interval $(c-\delta, c+\delta)$ below by $f(x)-\epsilon$ and above by $f(x)+\epsilon$.

Solution:
f is continuous in an open interval containing c. So there exists a $\delta > 0$ such that if $0 < |x-c| < \delta$, then $|f(x) - f(c)| < \epsilon$.
So if

$$x \in (c-\delta, c+\delta)$$

then

$$f(x) \in (f(c)-\epsilon, f(c)+\epsilon)$$

Upon inspection, we have the desired result.

□

7. This exercise outlines the proof of the Extreme Value Theorem. Prove the first part, that if function f is defined and continuous on interval $[a,b]$, then f achieves a maximum on $[a,b]$, by carrying out the following steps.
b) Let

$$g(x) = \frac{1}{M - f(x)}$$

If f never takes on the value M on $[a,b]$, explain why g is bounded on $[a,b]$.

Solution:
If f is never equal to M, then $f(x) - M$ is never 0. So $g(x)$ is continuous on $[a,b]$. Therefore $g(x)$ must be bounded on $[a,b]$.

□

d) Show that $g(x) > B$ for some $x \in [a,b]$.

Solution:

$$g(x) = \frac{1}{M - f(x)} > B \Leftrightarrow \frac{1}{B} > M - f(x)$$

By part c), $M - \frac{1}{B}$ is not an upperbound for R_f. So there exists $x_0 \in [a,b]$ such that $f(x_0) > M - \frac{1}{B}$. Clearly x_0 is the desired point.

□

11. Explain why the above version of the Intermediate Value Theorem implies the version in the text.

Solution:
If f be defined and continuous on $[a, b]$, and let v be between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = v$.
Let $a', b' \in [a, b]$ such that $f(a')$ is the Maximum value of f and $f(b')$ is the Minimum value of f. By symmetry we will assume that $a' < b'$. By problem 13 below we see that f is defined and continuous on $[a', b']$. So by the above, for v between $f(a')$ and $f(b')$ there exists $c \in (a', b')$ such that $f(c) = v$.

□

13. Show that if function f is continuous on (a, b) and $a < c < d < b$ then f is continuous on $[c, d]$

Solution:
Let $e \in (c, d)$.

$$\lim_{x \to e} f(x) = f(e)$$

$\Rightarrow f(x)$ is continuous on (c, d).

$$\lim_{x \to c} f(x) = \lim_{x \to c+} f(x) = f(c)$$

and similarly

$$\lim_{x \to d-} f(x) = f(d)$$

So $f(x)$ is continuous on $[c, d]$.

□

17b. Verify that the given function has a zero in the given interval. Then apply the Bisection Method four times to obtain a shorter interval containing the zero.

$$f(x) = x - \cos x, [0, 1]$$

Solution:
$f(0) = -1$ and $f(1) = 1 - \cos 1 = 0.459698 > 0$. By the Intermediate Value Theorem $f(x)$ has a zero on the interval.

1: $c = \frac{1}{2}$, $f(c) = \frac{1}{2} - \cos\frac{1}{2} = -0.377583$
$f(c)f(a) > 0$, So our new interval is $[\frac{1}{2}, 1]$.

2: $c = \frac{3}{4}$, $f(c) = 0.0183111$
$f(c)f(a) < 0$, So our new interval is $[\frac{1}{2}, \frac{3}{4}]$

3: $c = \frac{5}{8}$, $f(c) - 0.185963$
$f(c)f(a) > 0$, So our new interval is $[\frac{5}{8}, \frac{3}{4}]$

4: $c = \frac{11}{16}$, $f(c) = -0.0853349$
$f(c)f(a) > 0$, So our new interval is $[\frac{11}{16}, \frac{3}{4}]$

□

18. Starting with an interval of length L, the Bisection Method is applied n times. How long is the resulting interval?

Solution:
$\frac{L}{2^n}$

□