Calculus Solutions: Chapter 2.6

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1. Show that each of the following functions has a removable discontinuity at c. Sketch the graph and show how to define or redefine the function so that it will be continuous at c.

b) $f(x) = \frac{x^2-9}{x-3}, c = 3$

Solution:

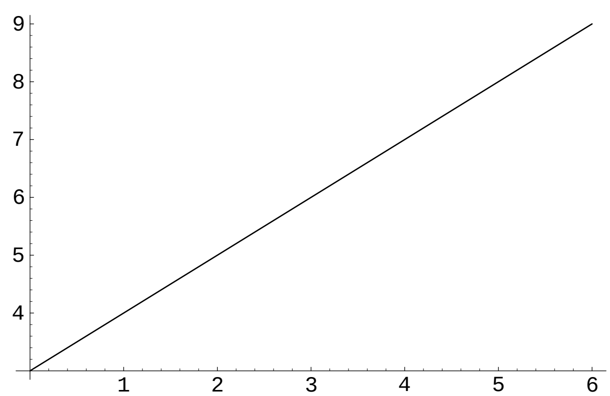

Figure 1: $\frac{x^2-9}{x-3}$ for $x \in [0, 6]$

Note that 3 is an isolated non domain point of f. We must show that $\lim_{x\to 3} \frac{x^2-9}{x-3}$ exists.

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3}(x+3) = 6$$

So $f(x)$ has a removable discontinuity at $c = 3$. Define a function $g(x) \equiv f(x)$ for $x \neq 3$ and $g(3) = 6$. $g(x)$ is continuous at 3.

□

d) $f(x) = \llcorner \sin x \lrcorner, c = \pi/2$ ($\llcorner \lrcorner$ is the floor function)

Solution:
Note that $\frac{\pi}{2}$ is an interior point in the domain of f.

$$\lim_{x \to \frac{\pi}{2}} f(x) = 0 \neq 1 = f(\frac{\pi}{2})$$

So $f(x)$ has a removable discontinuity at $c = \frac{\pi}{2}$. Define a function $g(x) \equiv f(x)$ for $x \neq \frac{\pi}{2}$ and $g(\frac{\pi}{2}) = 0$. $g(x)$ is continuous at $\frac{\pi}{2}$.

□

2. Show that each of the following functions has a jump discontinuity at c. Sketch the graph.
b) $f(x) = \llcorner x \lrcorner, c = n$, an integer

Solution:

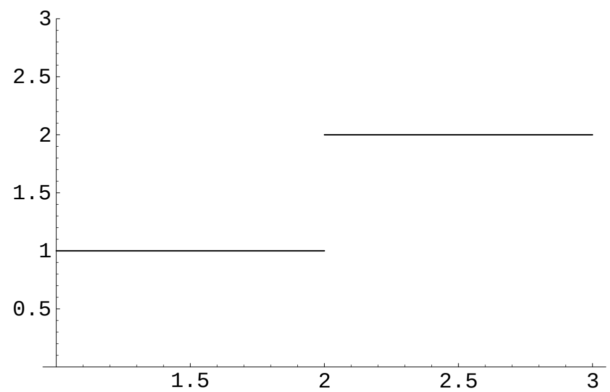

Figure 2: $\llcorner x \lrcorner$ for $x \in [1, 3]$

We must show that the two one-sided limits exist at n, but have different values.

$$\lim_{x \to n^-} f(x) = n - 1 \neq n = \lim_{x \to n^+} f(x)$$

$f(x)$ has a jump discontinuity at n.

□

d) $f(x) = \lfloor \sin x \rfloor, c = \pi$

Solution:

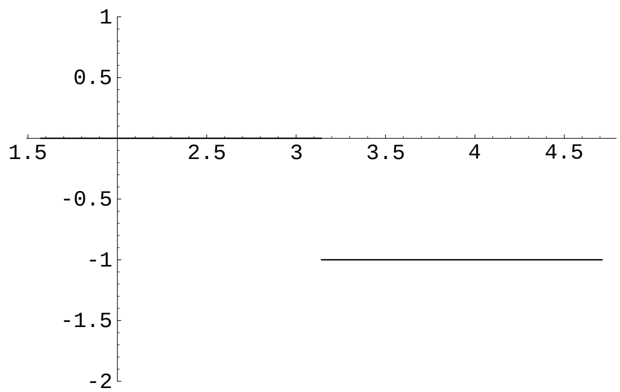

Figure 3: $\lfloor \sin x \rfloor$ for $x \in [\frac{\pi}{2}, \frac{3\pi}{2}]$

$$\lim_{x \to \pi^-} f(x) = 0 \neq -1 = \lim_{x \to \pi^+} f(x)$$

$f(x)$ has a jump discontinuity at π.

□

3. Show that each of the following functions has an infinite discontinuity at c. Sketch the graph.

b) $f(x) = \frac{1}{x-3}, c = 3$

Solution:

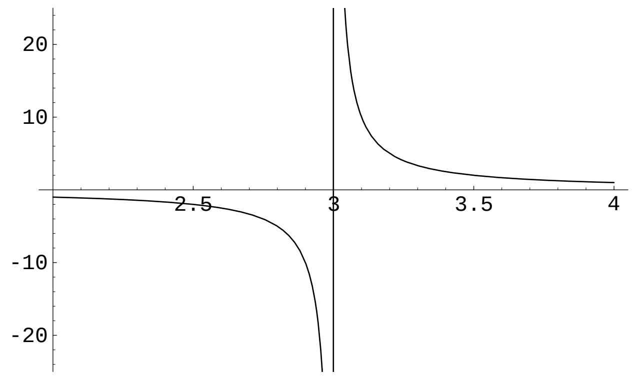

Figure 4: $\frac{1}{x-3}$ for $x \in [2, 4]$

We must show that one of the one sided limits is infinite.

$$\lim_{x \to 3^+} \frac{1}{x-3} = \infty$$

So $f(x)$ has an infinite discontinuity at 3.

$\square$

d) $f(x) = \ln x, c = 0$

Solution:

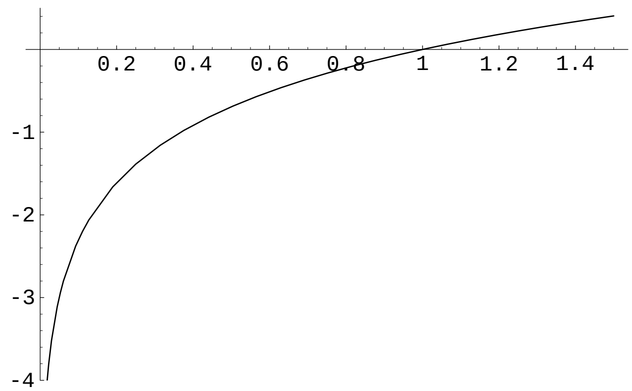

Figure 5: $\ln x$ for $x \in [0, 1.5]$

$$\lim_{x \to 0^+} \ln x = -\infty$$

So $f(x)$ has an infinite discontinuity at 0.

□

4. Determine whether the given function is continuous at c. Sketch the graph and classify any points of discontinuity.
e) $f(x) = \llcorner \cos x \lrcorner, c = \frac{\pi}{2}$

Solution:

$$\lim_{x \to \frac{\pi}{2}^-} \llcorner \cos x \lrcorner = 0 \neq -1 = \lim_{x \to \frac{\pi}{2}^+} \llcorner \cos x \lrcorner$$

So $f(x)$ is not continuous at $c = \frac{\pi}{2}$.

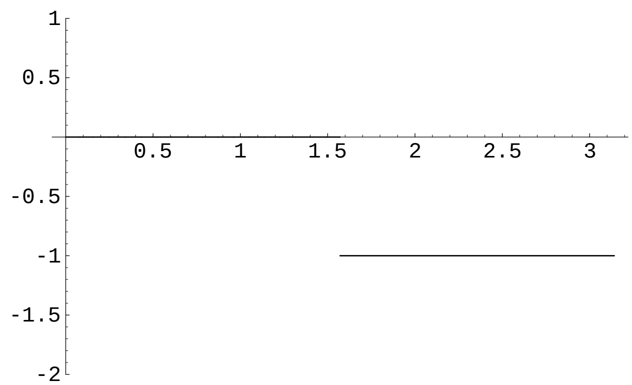

Figure 6: $\llcorner \cos x \lrcorner$ for $x \in [0, 1.5]$

Consider the above graph. $f(x)$ has a jump discontinuity at $c = \frac{\pi}{2}$.

$\square$

f) $f(x) = \llcorner \cos x \lrcorner, c = 0$

Solution:

$$\lim_{x \to 0^-} \llcorner \cos x \lrcorner = 0 = \lim_{x \to 0^+} \llcorner \cos x \lrcorner \neq 1 = \llcorner \cos 0 \lrcorner$$

So $f(x)$ is not continuous at $c = 0$.
$f(x)$ has a jump discontinuity at $c = 0$.

□

5. Determine the largest set of real numbers on which the function is continuous. Sketch the graph and classify any points of discontinuity.
b) $f(x) = \frac{1}{x^2+2x+1}$

Solution:
$f(x)$ is continuous everywhere except where it is undefined. $f(x)$ is undefined where $x^2 + 2x + 1 = 0$. The largest set of real numbers on which f is continuous is

$$(-\infty, -1) \bigcup (-1, \infty)$$

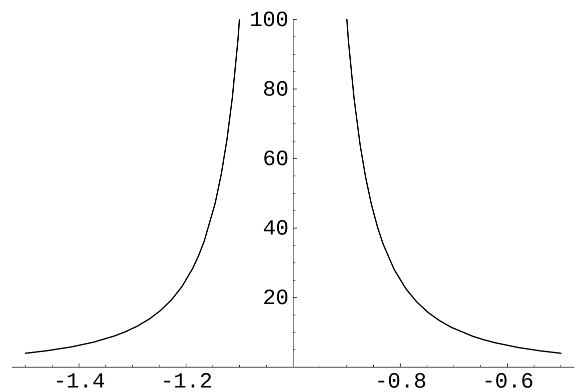

Figure 7: $\frac{1}{x^2+2x+1}$ for $x \in [-1.5, -.5]$

Consider the above graph. f has an infinite discontinuity at $c = -1$.

□

d) $f(x) = 4 - x^2$ for $x < 0$ and $f(x) = x + 4$ for $x > 0$.

Solution:

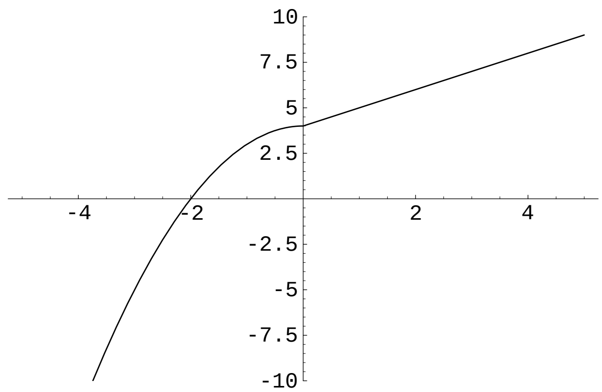

Figure 8: $f(x)$ for $x \in [-5, 5]$

Clearly the largest set of numbers on which f is continuous is

$$(-\infty, 0) \bigcup (0, \infty)$$

$\square$

f) $f(x) = \frac{1}{\sqrt{4-x^2}}$

Solution:
$f(x)$ is continuous if and only if $4 > x^2$. So the largest set of real numbers on which f is continuous is

$$(-2, 2)$$

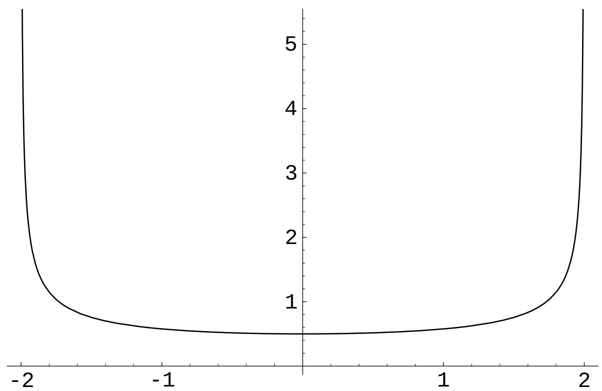

Figure 9: $f(x)$ for $x \in [-2, 2]$

Consider the above graph. $f(x)$ has an infinite discontinuity at $x = 2$ and $x = -2$.

$\square$

11. Show that the function $f(x) = \sin \frac{1}{x}$ has an essential discontinuity at 0 that is neither a jump discontinuity nor an infinite discontinuity.

Solution:
First, since 0 is an isolated non domain point of f, we must show that $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist. We will do this by showing that $\lim_{x \to 0+} \sin \frac{1}{x}$ does not exist. Suppose the limit exists and is equal to L. Let $\epsilon = \frac{1}{2}$. For any $\delta > 0$, we may choose a point $a \in (0, \delta)$ such that $\sin \frac{1}{a} = 1$ and we may choose a point $b \in (0, \delta)$ such that $\sin \frac{1}{b} = -1$. If

$$|\sin \frac{1}{a} - L| < \frac{1}{2}$$

then

$$1 \in (L - \frac{1}{2}, L + \frac{1}{2})$$

and if

$$|\sin \frac{1}{b} - L| < \frac{1}{2}$$

then

$$-1 \in (L - \frac{1}{2}, L + \frac{1}{2})$$

This is a contradiction. So $\lim_{x \to 0+} \sin \frac{1}{x}$ does not exist.
From the above we know that $f(x)$ is not continuous at 0 and that $f(x)$ does not have a jump discontinuity at 0. Noting that $|\sin \frac{1}{x}| \leq 1$ we determine that $f(x)$ does not have an infinite discontinuity at 0.

□