Calculus Solutions: Chapter 2.4

Aaron Peterson, Stephen Taylor

September 22, 2006

1. Find the limits, if they exist.

b) $\lim_{x\to\pi/6} \tan x$

Solution:
Since $\tan x$ is defined at $x = \frac{\pi}{6}$, we can determine the limit by evaluating at $\frac{\pi}{6}$.

$$\lim_{x\to\pi/6} \tan x = \tan(\frac{\pi}{6}) = \frac{\sqrt{3}}{3}$$

□

d) $\lim_{x\to 2\pi/3} \csc x$

Solution:
By the same method as in the previous problem, we determine that

$$\lim_{x\to 2\pi/3} \csc x = \csc(\frac{2\pi}{3}) = \frac{2\sqrt{3}}{3}$$

□

f) $\lim_{x\to 0} \frac{2x}{\tan x}$

Solution:

$$\lim_{x\to 0} \frac{2x}{\tan x} = \lim_{x\to 0} \frac{2x\cos x}{\sin x} = \lim_{x\to 0} 2\cos x \frac{1}{\frac{\sin x}{x}} = 2(1)(1) = 2$$

□

h) $\lim_{x\to 0} \frac{\tan 2x}{\sin 3x}$

Solution:

$$\lim_{x\to 0} \frac{\tan 2x}{\sin 3x} = \frac{\tan(x+x)}{\sin(2x+x)} = \lim_{x\to 0} \frac{\frac{2\tan x}{1-\tan^2 x}}{\sin 2x \cos x + \cos 2x \sin x}$$

$=$

$$\lim_{x\to 0} \frac{\frac{2\tan x}{1-\tan^2 x}}{2\sin x\cos^2 x + 2\cos^2 x \sin x - \sin x} = \lim_{x\to 0} \frac{\frac{2\cos x}{1-\tan^2 x}}{4\cos^2 x - 1} = \frac{2}{3}$$

Making use of the identities $\cos 2x = 2\cos^2 x - 1$ and $\sin 2x = 2\cos x \sin x$

□

j) $\lim_{x\to \pi/4} \frac{\tan(\pi-4x)}{x-\pi/4}$

Solution:

$$\lim_{x\to \pi/4} \frac{\tan(\pi-4x)}{x-\pi/4} = -4 \lim_{x\to \pi/4} \frac{\tan(\pi-4x)}{\pi-4x} = -4 \lim_{x\to 0} \frac{\tan x}{x} = -4(1) = -4$$

□

2. Find the limits, if they exist.
b) $\lim_{x\to 0} |x| \cos x$

Solution:
$\cos x$ is bounded near 0, and $\lim_{x\to 0} |x| = 0$. Applying Theorem 30 we determine that $\lim_{x\to 0} |x| \cos x = 0$.

□

d) $\lim_{x\to \pi}(x^3 - 5x^2 + 3)\sin x$

Solution:
Similar to part b), $\lim_{x\to \pi}(x^3 - 5x^2 + 3)\sin x = 0$.

□

f) $\lim_{x\to 0+} \cos\sqrt{x}$

Solution:
$\lim_{x\to 0+} \sqrt{x} = 0$ and $\lim_{x\to 0} \cos x = 1$. So we apply Theorem 27 to determine that

$$\lim_{x\to 0+} \cos\sqrt{x} = 1$$

□

h) $\lim_{x\to 4} \sqrt{\sin x}$

Solution:
Notice that $\sin 4 < 0$. Therefore $\sqrt{\sin 4}$ is not defined, so the limit does not exist.

□

4. Find the limits, if they exist.
b) $\lim_{x\to 2} e^{-x}$

Solution:

$$\lim_{x\to 2} e^{-x} = \frac{1}{\lim_{x\to 2} e^{x}} = \frac{1}{e^2} = e^{-2}$$

□

d) $\lim_{x\to 2} \frac{x+2}{e^x - e^2}$

Solution:
Since $\lim_{x\to 2}(x+2) = 4 \neq 0$ and

$$\lim_{x\to 2}(e^x - e^2) = e^2 - e^2 = 0$$

$\lim_{x\to 2} \frac{x+2}{e^x - e^2}$ does not exist.

□

f) $\lim_{x\to 3} 10^{-x}$

Solution:
$\lim_{x\to 3} 10^{-x} = 10^{-3}$ by Theorem 34.

□

7. Let f and g be two functions such that

$$\lim_{x\to 0} \frac{f(x)}{x} = 1$$

and

$$\lim_{x\to 0} \frac{g(x)}{x} = 0$$

.

What can you say about the following limits?
a) $\lim_{x\to 0} f(x)$

Solution:
Since $\lim_{x\to 0} \frac{f(x)}{x} = 1$ and $\lim_{x\to 0} x = 0$, we know that

$$\lim_{x\to 0} f(x) = 0$$

□

b) $\lim_{x\to 0} g(x)$

Solution:
It is clear that $\lim_{x\to 0} g(x) = 0$

□

c) $\lim_{x\to 0} \frac{g(x)}{f(x)}$

Solution:

$$\lim_{x\to 0} \frac{g(x)}{f(x)} = \frac{0}{1} = 0$$

□

d) $\lim_{x\to 0} \frac{g(x)}{xf(x)}$

Solution:
Let $g(x) \equiv 0$ and $f(x) \equiv 1$.

$$\lim_{x\to 0} \frac{g(x)}{xf(x)} = 0$$

Now let $g(x) = x$ and $f(x) = 1 + x$.

$$\lim_{x\to 0} \frac{g(x)}{xf(x)} = \lim_{x\to 0} \frac{1}{1+x} = 1$$

Since in both cases our functions satisfy the above requirements, we see that we cannot determine the value of $\lim_{x\to 0} \frac{g(x)}{xf(x)}$ with the information given.

□

e) $\lim_{x\to 0} \frac{g(x)}{x^2}$

Solution:
Let $g(x) \equiv 0$

$$\lim_{x\to 0} \frac{g(x)}{x^2} = 0$$

Now let $g(x) = x^2$

$$\lim_{x\to 0} \frac{g(x)}{x^2} = 1$$

Since in both cases our functions satisfy the above requirements, we see that we cannot determine the value of $\lim_{x\to 0} \frac{g(x)}{x^2}$ with the information given.

□

10. Let f be a function such that $f(x) > x^3$ for all x, and let g be a function such that $x^3 \leq g(x) \leq f(x)$ for all x. Given that

$$\lim_{x\to 0} f(x) = 0, \lim_{x\to 1} f(x) = 1, \lim_{x\to 2} f(x) = 8$$

and

$$\lim_{x\to -1} f(x) = 0$$

find the following if possible.

b) $\lim_{x\to 1} g(x)$

Solution:

$$1 = \lim_{x\to 1} x^3 \leq \lim_{x\to 1} g(x) \leq \lim_{x\to 1} f(x) = 1$$

so

$$\lim_{x\to 1} g(x) = 1$$

□

d) $\lim_{x\to -1} g(x)$

Solution:

$$-1 = \lim_{x\to -1} x^3 \leq \lim_{x\to -1} g(x) \leq \lim_{x\to -1} f(x) = 0$$

so we cannot determine $\lim_{x\to -1} g(x)$ from the information given.

□

11. Tell whether the given function is bounded on the given interval. If so, give a number B such that $|f(x)| \leq B$ for all x in the interval.
b) $f(x) = \frac{1}{x}$, (1,2)

Solution:
Every x in (1,2) is positive and larger than 1, so $f(x) = \frac{1}{x}$ decreases as x increases on the interval. $|f(x)| < 1$ for all x in (1,2).

□

d) $f(x) = \tan x$, (1,2)

Solution:
$\frac{\pi}{2}$ is in the interval (1,2). As x approaches $\frac{\pi}{2}$ from the left side, $\tan x$ grows without bound in the positive direction. As x approaches $\frac{\pi}{2}$ from the right side, $\tan x$ grows without bound in the negative direction. $\tan x$ is not bounded on (1,2).

□

f) $f(x) = \frac{1}{x^2+1}$, $(-\infty, \infty)$

Solution:
Similar to part b), $f(x) = \frac{1}{x^2+1}$ is decreasing as $|x|$ increases. $|f(x)| \leq 1$ for all x in $(-\infty, \infty$.

□