Calculus Solutions: Chapter 1.9

Aaron Peterson, Stephen Taylor

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Find values (exact if possible) for

1b. $\cos^{-1}\frac{\sqrt{3}}{2}$

Solution:

$$\cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$$

$\square$

1d. $\cos^{-1}\frac{-\sqrt{3}}{2}$

Solution:

$$\cos^{-1}\frac{-\sqrt{3}}{2} = \frac{5\pi}{6}$$

$\square$

1f. $\cot^{-1}(\sin\frac{\pi}{3})$

Solution:

$$\cot^{-1}\left(\sin\frac{\pi}{3}\right) = \cot^{-1}\frac{\sqrt{3}}{2} \approx 0.857072$$

$\square$

1h. $\cosh 0$

Solution:

$$\cosh 0 = \frac{e^0 + e^{-0}}{2} = 1$$

$\square$

2. Show that for any $x \in [-1, 1]$ that $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

Solution:
Applying (1.33) to find

$$x = \cos(\cos^{-1}(x)) = \sin\left(\frac{\pi}{2} - \cos^{-1}(x)\right)$$

Taking the inverse sine of both sides and rearranging, we find

$$\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$$

$\square$

Solve the equations, giving all real solutions.

6c. $\sin x = -0.5$

Solution:
Taking the inverse sine of each side we find

$$x = \sin^{-1}(-0.5) = -\frac{\pi}{6}$$

Thus the solution set is $\left\{ -\frac{\pi}{6} + n\pi \middle| n \in \mathbb{Z} \right\}$.

$\square$

6e. $\sin 2x = \cos x$

Solution:
Using the double angle identity $\sin 2x = 2 \cos x \sin x$ the above equation becomes

$$2 \cos x \sin x = \cos x \Rightarrow \sin x = \frac{1}{2}$$

Taking the inverse sine we find $x = \frac{\pi}{6}$. Thus the solution set is $\left\{ \frac{\pi}{6} + n\pi \middle| n \in \mathbb{Z} \right\}$.

$\square$

8a. Solve the following equation showing all real solutions

$$\sin 2x = \cos 2x$$

Solution:
Dividing both sides of the equation by $\cos 2x$ we find the new equation

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x} = 1 \Rightarrow \tan^2 x + 2 \tan x - 1 = 0$$

Applying the quadratic formula, we find

$$\tan x = -1 \pm \sqrt{2}$$

Thus the solution set is $\{n\pi - 1 \pm \sqrt{2} | n \in \mathbb{Z}\}$.

$\square$

9. The principal domain of the secant function should definitely include $[0, \frac{\pi}{2})$, but there are choices for another piece that will include the entire range of the secant function. Many choose the second quadrant, but since the secant function is also increasing there as it is in the first quadrant, we can avoid difficulties later by choosing a decreasing portion. We will choose the principal domain $[-\pi, -\frac{\pi}{2}) \cup [0, \frac{\pi}{2})$. All of the reasons for doing so will only become apparent in Chapters 3 and 7. We can now define the inverse secant function by

$$y = \sec^{-1} x \text{ iff } x = \sec y \text{ and } y \in \left[-\pi, -\frac{\pi}{2} \right) \cup \left[0, \frac{\pi}{2} \right)$$

The graph of $y = \sec^{-1} x$ is shown in Figure 1.68. An alternative notation for $\sec^{-1} x$ is arcsec$x$. Find the exact values of the following:

a) $\sec^{-1} 2$
b) $\sec^{-1} \frac{-2}{\sqrt{3}}$
c) $\sec^{-1}(\csc \frac{\pi}{3})$
d) $\sec^{-1}(\tan \frac{3\pi}{4})$

Solution:

$$\sec^{-1} 2 = \frac{\pi}{3}$$

$$\sec^{-1} \frac{-2}{\sqrt{3}} = \frac{5\pi}{6}$$

$$\sec^{-1} \left(\csc \frac{\pi}{3} \right) = \sec^{-1} \left(\frac{2}{\sqrt{3}} \right) = \frac{\pi}{6}$$

$$\sec^{-1} \left(\tan \frac{3\pi}{4} \right) = \sec^{-1}(-1) = \pi$$

$\square$

Show the following identities:

12a.

$$\cot^{-1} x = \tan^{-1} \frac{1}{x} \quad \text{if } \; x > 0$$
$$\cot^{-1} x = \pi + \tan^{-1} \frac{1}{x} \quad \text{if } \; x < 0$$

Solution:
We note the first identity holds for all x, so we prove just it:

$$\frac{1}{x} = \frac{1}{\cot(\cot^{-1}(x))} = \tan(\cot^{-1}(x))$$

Taking the inverse tangent of both sides of the above, we find

$$\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}(x)$$

$$\square$$

12b.

$$\sec^{-1} x = \cos^{-1} \frac{1}{x} \quad \text{if } \; x > 0$$
$$\sec^{-1} x = -\cos^{-1} \frac{1}{x} \quad \text{if } \; x < 0$$

Solution:
We note the first identity holds for all x, so we prove just it:

$$\frac{1}{x} = \frac{1}{\sec(\sec^{-1}(x))} = \cos(\sec^{-1}(x))$$

Taking the inverse cosine of both sides of the above, we find

$$\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}(x)$$

$$\square$$

12c.

$$\csc^{-1} x = \sin^{-1} \frac{1}{x} \quad \text{if } x > 0$$

$$\csc^{-1} x = -\pi - \sin^{-1} \frac{1}{x} \quad \text{if } x < 0$$

Solution:
We note the first identity holds for all x, so we prove just it:

$$\frac{1}{x} = \frac{1}{\csc(\csc^{-1}(x))} = \sin(\csc^{-1}(x))$$

Taking the inverse sine of both sides of the above, we find

$$\sin^{-1}\left(\frac{1}{x}\right) = \csc^{-1}(x)$$

□

14. Derive the identity

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

by using

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

and computing the turn

$$\tan\left(2\tan^{-1}\frac{1}{5}\right), \ \tan\left(4\tan^{-1}\frac{1}{5}\right), \ \text{and } \ \tan\left(4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}\right)$$

Solution:
We first do the desired computations

$$\tan\left(2\tan^{-1}\frac{1}{5}\right) = \frac{2\tan(\tan^{-1}(1/5))}{1-\tan(\tan^{-1}(1/5))^2} = \frac{2/5}{24/25} = \frac{5}{12}$$

$$\tan\left(4\tan^{-1}\frac{1}{5}\right) = \frac{2\tan(2\tan^{-1}(1/5))}{1-\tan(2\tan^{-1}(1/5))^2} = \frac{2(5/12)}{119/144} = \frac{120}{119}$$

We note that this previous result is very close to 1, and $\tan(\pi/4) = 1$. Thus we use formula (1.50) to compute

$$\tan\left(4\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right) = \frac{\tan\left(4\tan^{-1}\left(\frac{1}{5}\right)\right) - 1}{1+\tan\left(4\tan^{-1}\left(\frac{1}{5}\right)\right)} = \frac{1/119}{239/119} = \frac{1}{239}$$

Taking the inverse tangent of the last equation and rearranging terms we find

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

as desired.

$\square$

17. Since $y = \sinh x$ is a one-to-one function, it has an inverse. If we define the inverse hyperbolic sine function by

$$y = \sinh^{-1} x \text{ iff } x = \sinh y$$

show that

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$$

Solution:
We solve the equation $y = \sinh x$ for x by the following:

$$y = \sinh x \equiv \frac{e^x - e^{-x}}{2}$$

Multiplying by a factor of $2e^y$ and rearranging we have

$$e^{2y} - 2xe^y - 1 = 0$$

Noting this is a quadratic equation in e^y, we find the solutions

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

Taking the positive part, we find

$$y = \ln\left(x + \sqrt{x^2+1}\,\right)$$

as desired.

$\square$