Calculus Solutions: Chapter 1.8

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Use the properties of logarithmic functions to simplify the following expressions.

1b. $\log_3(7/81)$

Solution:

$$\log_3(7/81) = \log_3 7 - \log_3 81 = \log_3 7 - 4$$

□

1d. $\ln(4e^{2x+3})$

Solution:

$$\ln(4e^{2x+3}) = \ln 4 + \ln(e^{2x+3}) = \ln 4 + 2x + 3$$

□

1f. $\ln((1+x)e^{-3x})$

Solution:

$$\ln((1+x)e^{-3x}) = \ln(1+x) + \ln(e^{-3x}) = \ln(1+x) - 3x$$

□

Use the properties of logarithmic functions to solve for x in each of the following equations.

3b. $3^x = 5^{2x-1}$

Solution:
Taking the logarithm base 3 of both sides of the equation, we have

$$x = \log_3(5^{2x-1}) = (2x-1)\log_3 5$$

Solving for x we find

$$x = \frac{\log_3 5}{2\log_3 5 - 1}$$

□

3d. $5^x 4^{x+1} = 20^x$

Solution:
Taking the logarithm base 5 of both sides of the equation, we have

$$x + (x+1)\log_5 4 = x(\log_5 20)$$

Thus

$$\frac{x+1}{x} = \frac{\log_5(20) - 1}{\log_5 4} \Rightarrow \frac{1}{x} = \frac{\log_5(20) - \log_5(4) - 1}{\log_5 4} = \frac{1-1}{\log_5 4} = 0$$

Thus no solution exists for $x \in \mathbb{R}$. If we let $x \to -\infty$, we obtain a solution to the equation.

□

3f. $1 + \ln(3x) = 3 + \ln(x^2)$

Solution:
Rearranging we have

$$\ln(3x) - \ln x^2 = 2$$

Thus we obtain the equation

$$\ln\left(\frac{3}{x}\right) = 2 \Rightarrow \frac{3}{x} = e^2 \Rightarrow x = \frac{3}{e^2}$$

□

4. Prove that the growth rate k and the doubling time D of an exponential growth model are related by the equation $kD = \ln 2$.

Solution:
The doubling time D occurs when the the following equation is satisfied

$$2Q_0 = Q_0 e^{kD} \Rightarrow D = \frac{\ln 2}{k}$$

Thus we find

$$Dk = \ln 2$$

□

7. If $Q = Q_0 e^{kt}$ is an exponential growth model with doubling time D, show that the model may be written as $Q = Q_0 2^{t/D}$.

Solution:
Using the previous result, we have

$$Q = Q_0 \exp(kt) = Q_0 \exp\left(\left(\frac{\ln 2}{D}\right)t\right) = \exp\left(\ln 2^{t/D}\right) = 2^{t/D}$$

as desired.

$\square$

9. Suppose the rabbit population of Tulare County grows at an exponential rate. If the number of rabbits in 1985 was estimated at 30,000 and the number of rabbits in 1988 was estimated at 35,000, about what would the rabbit population have been in 1993?

Solution:
We take year 0 to be 1985 and use the two data points $(0, 3 \cdot 10^4)$ and $(3, 3.5 \cdot 10^4)$ to determine the two constants A and b of the exponential function $y = Ab^x$ and find

$$y = 3 \cdot 10^4 (1.05273)^x$$

Extrapolating, we estimate the rabbit population in 1993 to be

$$1993 \to 3 \cdot 10^4 (1.05273)^8 \approx 45254$$

$\square$

11. Find the future value of \$100 in five years at 12% compounded continuously.

Solution:
The future value is given by

$$FV = 100e^{0.12 \cdot 5} = 182.2$$

$\square$

13. Find the present value of a government savings bond that will be worth \$1000 in six years; its earning rate is 4.75% compounded continuously.

Solution:
We solve the equation

$$1000 = PVe^{0.0475} \Rightarrow PV = 1000e^{-0.285} \approx 752.01$$

$\square$

14b. Carbon-14 is a radioactive isotope of carbon that is created in the upper atmosphere through the bombardment of nitrogen by cosmic rays. The half-life of C-14 is 5730 years. The process of creation of C-14 has been going on long enough that we assume the proportion of C-14 is a constant fraction of all the carbon in the atmosphere. All plants take carbon dioxide from the atmosphere, so that in living plants the proportion of C-14 is the same as in the atmosphere. All other living things eat plants or other animals, so that in all living tissue the proportion of C-14 is the same as in the atmosphere. When an organism dies, the C-14 in its tissue decays, so that by measuring the proportion of C-14 remaining we can estimate when the organism died. Independent means of measuring age have verified that C-14 dating is fairly accurate up to ages of about 3000 years.

b)A newly-cut tree is known by counting its rings to be 2615 years old. What percentage of the original C-14 should its core contain?

Solution:
The fraction of CO2 is given by $2^{-t/T}$, where T is the half life, and t is the time of interest. Thus the fraction remaining is

$$2^{-2615/5730} \approx .7288$$

□

23. Uranium-238 ultimately decays into lead, with a half-life of 4.5×10^9 years. Because of its low melting point, lead is removed from molten lava, so that a fresh lava flow contains no lead. The amount of lead present in lava therefore indicates the age of the lava flow. If a certain lava sample contains four atoms of lead for every 100 atoms of U-238, how long ago was the volcanic eruption that produced the lava?

Solution:
Using the formula given in the previous problem, we obtain the equation

$$.96 = 2^{(-t/(4.5 \cdot 10^9))} \Rightarrow t = 2.54626 \cdot 10^9$$

□

Use a calculator to find a three-place estimate for each of the following values.

29b. $\ln 10$

Solution:

$$\ln 10 \approx 2.302585093$$

□

29d. $\log_5 3$

Solution:

□

$$\log_5 3 \approx 0.6826061945$$

29e. $1.7^{3.5}$

Solution:

$$1.7^{3.5} \approx 6.4057682833$$

□