Calculus Solutions: Chapter 1.6

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Find formulas for $f+g$, $f-g$, fg, f/g, $g \circ f$, and $f \circ g$ for each of the following pairs of functions.

1b. $f(x) = 1 - x$, $g(x) = \frac{1}{x}$

Solution:

$$f + g = \frac{1}{x} + 1 - x$$
$$f - g = -\frac{1}{x} + 1 - x$$
$$fg = (1-x)\frac{1}{x} = \frac{1}{x} - 1$$
$$f/g = \frac{1-x}{1/x} = x(1-x) = x - x^2$$
$$g \circ f = \frac{1}{1-x}$$
$$f \circ g = 1 - \frac{1}{x}$$

□

1d. $f(x) = e^x$, $g(x) = \sin x$

Solution:

$$f + g = e^x + \sin x$$
$$f - g = e^x - \sin x$$
$$fg = e^x \sin x$$
$$f/g = \frac{e^x}{\sin x} = e^x \csc x$$
$$g \circ f = \sin(e^x)$$
$$f \circ g = e^{\sin x}$$

$\square$

1h. $f(x) = \frac{1}{x+2}$, $g(x) = \frac{x}{x-1}$

Solution:

$$f + g = \frac{1}{x+2} + \frac{x}{x-1} = \frac{x^2 + 3x - 1}{(x+2)(x-1)}$$

$$f - g = \frac{1}{x+2} - \frac{x}{x-1} = -\frac{x^2 + x + 1}{(x+2)(x-1)}$$

$$fg = \frac{x}{(x+2)(x-1)}$$

$$f/g = \frac{x-1}{x(x+2)}$$

$$g \circ f = \frac{\frac{1}{x+2}}{\frac{1}{x+2} - 1} = \frac{1}{1-(x+2)} = -\frac{1}{1+x}$$

$$f \circ g = \frac{1}{\left(\frac{x}{x-1}\right) + 2} = \frac{1}{\frac{x+2x-2}{x-1}} = \frac{x-1}{3x-2}$$

$\square$

If $f(x) = 3x + 2$, find expressions for the following:

2b. $\frac{1}{f(x)}$

Solution:

$$\frac{1}{f(x)} = \frac{1}{3x+2}$$

$\square$

2d. $f\left(1 - \frac{1}{f(x)}\right)$

Solution:

$$f\left(1 - \frac{1}{f(x)}\right) = f\left(1 - \frac{1}{3x+2}\right) = f\left(\frac{3x+1}{3x+2}\right)$$

$$= 3\left(\frac{3x+1}{3x+2}\right) + 2 = \frac{15x+7}{3x+2}$$

$\square$

Given the graphs of f and g in Figure 1.46, sketch graphs of the following

3b. $f - g$

Solution:

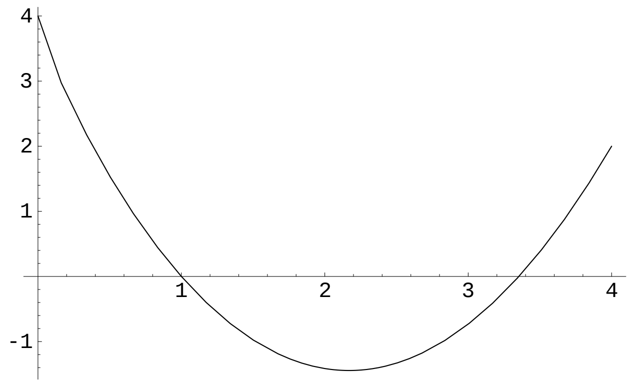

Figure 1: $f(x) - g(x)$ for $x \in [0, 4]$.

□

3d. f/g

Solution:

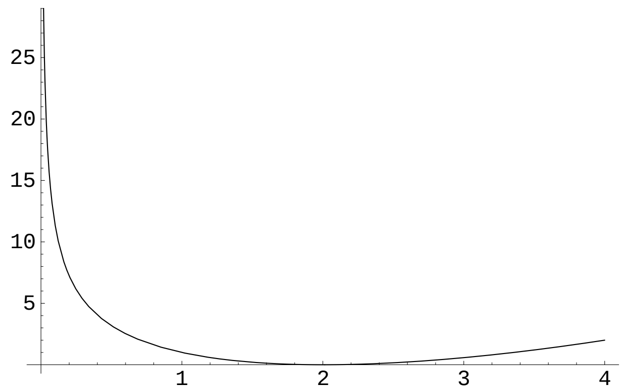

Figure 2: $f(x)/g(x)$ for $x \in [0, 4]$.

□

3f. $f \circ g$

Solution:

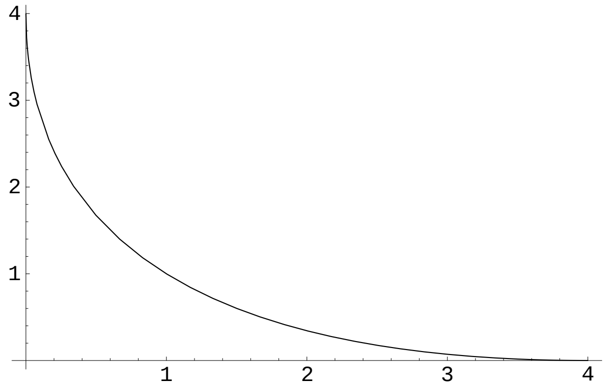

Figure 3: $f \circ g = f(g(x))$ for $x \in [0, 4]$.

5. A manufacturer can sell quantity $q = 50 - p$ units of his product when the price is p dollars per unit, thereby gaining a revenue of $R = pq$ dollars. What should the price be for maximum revenue?

Solution:
Substituting the quantity into the revenue function, we find

$$R = pq = p(50 - p) = -p^2 + 50p$$

We seek the maximum value of this graph. We suspect it occurs at $p = 25$. To show this, we define a new number $r = 25 + \epsilon$ where $\epsilon \in \mathbb{R}$ and is also taken to be a "very small" number. Substituting into the equation we find

$$\begin{aligned} R(25 + \epsilon) = -(25 + \epsilon)^2 + 50(25 + \epsilon) &= -25^2 - 50\epsilon - \epsilon^2 + 1250 + 50\epsilon \\ &= 625 - \epsilon^2 \end{aligned}$$

Thus when $\epsilon = 0$, we have obtained the desired maximum (Note it $\epsilon \neq 0$ $R(p)$ is forced to decrease).

$\square$

8. Find a textbook on elementary functions and look up and state the following properties of polynomial functions.

a) The Division Algorithm
b) The Remainder Theorem
c) The Factor Theorem

Solution:

The division algorithm may be found here:
http://web.usna.navy.mil/ wdj/book/node64.html

The Remainder Theorem may be found here:
http://mathworld.wolfram.com/PolynomialRemainderTheorem.html

The Factor Theorem may be found here:
http://mathworld.wolfram.com/PolynomialFactorTheorem.html

□

10. Find a polynomial function having exactly the zeros $-1, 1, 3$ and $\frac{3}{4}$.

Solution:
The desired polynomial is

$$f(x) = (x+1)(x-1)(x-3)\left(x - \frac{3}{4}\right) = x^4 - \frac{15}{4}x^3 + \frac{5}{4}x^2 + \frac{15}{4}x - \frac{9}{4}$$

□

12. Find all the zeros of the polynomial function
$f(x) = 2x^4 - 11x^3 + 2x^2 + 43x - 42$, given that $3/2$ and -2 are zeros.

Solution:
We first compute by long division

$$\frac{f(x)}{x+2} = 2x^3 - 15x^2 + 32x - 21$$

then we compute

$$\frac{f(x)}{(x+2)(x-3/2)} = 2x^2 - 12x + 14$$

Applying the quadratic formula we find the roots to be

$$x_{\pm} = \frac{12}{4} \pm \frac{\sqrt{144-112}}{4} = 3 \pm \sqrt{2}$$

Thus all the zeros are $\frac{3}{2}, -2, 3+\sqrt{2}, 3-\sqrt{2}$.

□