Chapter 7 Notes

Probability

7a: The Basics

Fundamentals

Definition 1. **Outcomes** are the most basic possible results of observations or experiments

Definition 2. An **event** consists of one or more outcomes that share a property of interest.

Definition 3. The probability of an event, expressed as $P(\text{event})$, is always between 0 and 1. A probability of 0 means the event is impossible and a probability of 1 means the event is certain.

Example 1. Suppose we toss a coin three times. The possible outcomes are

HHH, HHT, HTH, THH, HTT, THT, TTH, TTT

An example of an event would be the number of times only one head occurred.

$$P(\text{one head}) = 3/8 = .375$$

Determining Probabilities

If you wish to determine the probability of an event A of occurring in a distribution of outcomes, there is a method for doing so:

1: Count the total number of possible outcomes.

2: Among all the possible outcomes, count the number of ways the even of interest, A, can occur.

3: Determine the probability, $P(A)$, from

$$P(A) = \frac{\text{number of ways } A \text{ can occur}}{\text{total number of outcomes}}$$

Example 2. Calculate probability of rolling a 7 with a pair of dice.

Empirical Probability

Suppose that geologists have shown there is evidence, a 9.0 magnitude earthquake has hit the California area 4 times in the last thousand years. They might make the estimate that the probability of an 9.0 magnitude earthquake striking the California area has a probability of .004 for any given year.

This is an example of an **empirical probability**.

Example 3. Suppose you role a pair of dice an get 7 on 18 out of 23 roles. What is the empirical probability of rolling a 7? How does this compare to the theoretical probability. What conclusions can you make?

Definition 4. Suppose the probability of an even A is $P(A)$. Then the probability that the even A does not occur is $1 - P(A)$.

Example 4. What is the probability of not rolling a 7 on a pair of dice?

Probability Distributions

Definition 5. A **probability distribution** represents the probabilities of all possible events.

To make a table of a probability distribution:

1: List all possible outcomes.

2: Identify outcomes that represent the same event. Find the probability of each event.

3: Make a table in which one column lists each even and another column lists each probability. The sum of all the probabilities must be 1

Example 5. Make a probability distribution for all possible rolls of a pair of dice.

7b: Combining Probability

Types of events

Example 6. (Independent Event) Suppose you toss a coin in the air 50 times. Lets say you get heads the first 49 times. What is the probability of getting heads on the 50th time?

Example 7. (Dependent Event) Suppose you place 50 slips of paper in a box, each with a number on it 1 through 50. What is the probability of drawing the number 9.

Lets say you draw 25 pieces of paper out. If the number 9 was not drawn, what is the probability of drawing 9?

Definition 6. Two events are **independent** if the outcome of one does not affect the probability of the other event. If A and B are independent events we calculate the probability of A and B occurring together by the formula:

$$P(A \text{ and } B) = P(A) \times P(B)$$

If $A_1, A_2, \ldots A_n$ are n independent events, we calculate the probability of all the events occurring by the formula

$$P(A_1 \text{ and } A_2 \text{ and } \ldots\ A_n) = P(A_1) \times P(A_2) \times \cdots \times P(A_n)$$

Example 8. Suppose you have 5 dice. What is the probability of rolling the all and getting a six on each?

Definition 7. Two events are said to be **dependent** if the outcome of one even affects the probability of the other event. The probability that dependent events A and B occur together is

$$P(A \text{ and } B) = P(A) \times P(B \ given \ A)$$

Example 9. Suppose you put 3 red balls, 5 blue balls, and 2 black balls in a bag. If you draw out three balls.

What is the probability of drawing out a blue, then a black, then another blue?

What is the probability of drawing out a red each time?

What is the probability of drawing out a black, then another black, then a blue, then a red, then another red?

Definition 8. If A and B are independent events we calculate the probability of A or B occurring together by the formula:

$$P(A \text{ or } B) = P(A) + P(B)$$

If $A_1, A_2, \ldots A_n$ are n independent events, we calculate the probability of any of the events occurring by the formula

$$P(A_1 \text{ or } A_2 \text{ or } \ldots A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

Example 10. If you draw 1 card from a deck, what is the probability of getting a 4 or 8 or queen?

Definition 9. Two events are **overlapping** if they can occur together. If A and B are overlapping events, the probability that either A or B occurs is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and} B)$$

Example 11. If you draw 1 card from a deck, what is the probability of getting a 3 or a spade?

Example 12. If you draw 1 card from a deck, what is the probability of getting a 5 or a 7, or a spade, or a heart?

Definition 10. Suppose the probability of an event A occurring in one trial is $P(A)$. If all trials are independent, the probability that event A occurs as least once in n trials is

$$1 - P(\text{no events } A \text{ in } n \text{ trials}) = 1 - [P(\text{not } A \text{ in one trial})]^n$$

Example 13. If a major flu occurs every 54 years on average, what is the probability of a major flu occurring in the next 20 years?

The Law of Large Numbers

Definition 11. The **law of large numbers** applies to a process for which the probability of an event A is $P(A)$ and the results or repeated trials are independent. It states:

If the process if prepared through many trials, the proportion of the trials in which event A occurs will be closer to the probability $P(A)$. The larger the number of trials, the closer the proportion should be to $P(A)$.

Example 14. Lets saw you roll a die 10,000 times, a graph of the number of rolls plotted against the proportion of getting a three may look something like the following:

Expected Value and the Gamblers Fallacy

Expected value gives a measure of an overall average for a distribution of events with associated probabilities.

Example 15. Lets say you roll a die 10,000 times. What is the expected value for the average number you roll?

Example 16. Lets say you sell life insurance. You have sold 10000 policies. You now from historic information that the probability that one of your insurees will die is .007 per year. If the each policy is worth $100,000, how much do you expect to pay each year?

Historical Note: The French mathematician and Blaise Pascal used expected value to argue for believing in God. He used p to represent the probability that God exists, even if it is very small. He then assigned values for the consequences of believing and not believing in God. In particular, he claimed that the consequence of not believing in God is the case that God actually exists is infinitely bad (negative). Thus, regardless of how small p might be, the expected value of believing in God is positive and the expected value of not believing in God is infinitely negative. He concluded that one should believe in God.

Definition 12. The **gambler's fallacy** is the mistaken belief that a streak of bad luck makes a person "due" for a streak of good luck.

Gambling events are independent by law. Therefore there is no such thing a luck.

All gambling games give the house an **edge**. Stated another way, the house always has a higher probability of winning than you do. The closest edge is in the game of blackjack where your probability of winning is roughly 49.2% and the houses' is 50.8%.

Example 17. If there were one hundred million games of blackjack played in Las Vegas last year about how many did the casinos win, about how many did the players win? Assuming the average bet was 5 dollars, how much money did the casinos make off of blackjack?

Why do you think casinos give out free food/shows/etc. to people who gamble for long periods of time?

7e: Counting

Example 18. If license plates consist of seven symbols that can be either numbers or letters, how many unique license plates are there?

Arrangements with repetition:

Definition 13. If we make r selections from a group of n choices, a total of

$$n \times n \times n \times \cdots \times n = n^r$$

different arrangements are possible.

Example 19. If you put four slips of paper labelled 1 through 4 into a bag, how many ways different ways can you draw out the four slips if order matters?

Definition 14. Mathematically, we are dealing with **permutations** whenever all selections come from a single group of items, no item may be selected more than once, and the order of arrangement matters. The total number of permutations possible with a group of n items is $n!$.

Note

$$n! = n \times (n-1) \times (n-2) \cdots \times 2 \times 1$$

Suppose there are 10 people running for 4 positions of different political positions. The top vote getter receives the highest position, the second gets the second highest position, etc. How many ways can the 10 people receive the four positions?

Definition 15. If we make r selections from a group of n items, the number of permutations is

$$P_r^n = \frac{n!}{(n-r)!}$$

Example 20. How many ways can a basketball coach form a team of 5-players from 12 players?

Combinations are simply permutations where order does not matter.

Example 21. You decide to go on a trip and you can only take 5 of your 23 dresses. How many ways can you do this?

Example 22. You decide you want to have four different rims on your car. If the car store has 17 different types of rims to choose from, how many ways can you install rims?