Chapter 8 Notes

Probability

8a: Linear vs Exponential Growth

Definitions

Definiton 1. Linear growth occurs when a quantity grows by the same absolute amount in each unit of time.

Definiton 2. Exponential growth occurs when a quantity grows by the same relative amount–that is, by the same percentage– in each unit of time.

Example 1. 1) BYU has added 500 students per year to its student body for the past 20 years.

2) You receive a raise of 5% each year for 30 years.

3) The population of the United States has increased 2% per year for the past 100 years.

4) Graphical Examples

5) Doubling Example: One lucky day, you meet a leprechaun who promises to give you fantastic wealth, but hands you only a penny before disappearing. You head home and place the penny under your pillow. The next morning, to your surprise, you find two pennies under your pillow. The following morning, you find four pennies, and the fourth morning, eight pennies. Apparently, the leprechaun gave you a *magic* penny. While you sleep, each magic penny turns into two magic pennies.

How much money would you have after t days? How much money would you have in one month?

Facts about Exponential Growth

i) Exponential growth leads to repeated doublings. With each doubling, the amount of increase is approximately equal to the sum of all preceding doublings.

ii) Exponential growth cannot continue indefinitely. After only a relatively small number of doublines, exponentially growing quantities reach impossible proportions.

Doubling and Half-Life

Example 2. Suppose we have a population of 1,000 rabbits that doubles every month. How many rabbits would there be in one month? How many in 2? How many in 5?

Formula:

After a time t, an exponentially growing quantity with a doubling time of T increases in size by a factor of $2^{t/T}$. The new value of the growing quantity is related to its initial value by

$$\text{new value} = \text{initial value} \cdot 2^{t/T}$$

Example 3. Suppose the consumer price index doubles roughly every 20 years. How much will a 2005 dollar be worth 35 years from now?

Example 4. World population doubled from 3 billion in 1960 to 6 billion in 2000. Suppose that world population continues to grow with a doubling time of 40 years. What will the population be in 2030? 2200? 3000?

Exponential decay occurs whenever a quantity decreases by the same percentage in every fixed time period. If the quantity decreases by half, we call the decrease the **half-life** of the quantity.

Example 5. Suppose you bought 20 bags of potato chips. In two weeks, you only had ten bags left. The half life of your potato chip supply would be two weeks.

Example 6. Plutonium-239 has a half-life of approximately 24,000 years.

Formula

After a time t, an exponentially decaying quantity with a half-life of T decreases in size by a factor of $(1/2)^{t/T}$. The new value of the decaying quantity is related to its initial value by

$$\text{new value} = \text{initial value} \cdot \left(\frac{1}{2}\right)^{t/T}$$

Example 7. Carbon-14 has a half-life of about 5700 years, and it is known to collect in organism only while they are alive. Once they die, it only decays. What fraction of carbon-14 in an animal bone still remains 7000 years after the animal died?

Logarithms

A logarithm is simply the inverse of an exponential.

$\log_{10} x$ is the power to which 10 much be raised to obtain x. Put another way, it asks the question 10 to what power gives me x.

Example 8. i) What is $\log_{10}(10000)$

ii) What is $\log_{10}(1/(100))$

iii) What is $\log_{10}(6)$

Doubling and Half-Life Formulas

Define the variable r to be the **fractional growth rate** of a population expressed in terms of a decimal. For example if the growth rate of a population is 20%, then $r = .2$.

Formula

For an exponentially growing quantity with a fractional growth rate r, the doubling time is

$$T_2 = \frac{\log_{10} 2}{\log_{10}(1 + r)}$$

For an exponentially decaying quantity, r will be negative. Furthermore, the half life for a decaying populaiton is

$$T_{1/2} = -\frac{\log_{10} 2}{\log_{10}(1+r)}$$

Example 9. A population of bacteria is growing at a rate of 36% every minute. What is the doubling time of the population?

Example 10. Scientists believe the Earth once had naturally existing plutonium-239. Suppose the Earth had 10 trillion tons of Pu-239 when it formed. Given plutonium's half-life of 24,000 years and the Earth's current age of 4.6 billion years, how much would remain today?

Population Growth

The following graph represents the world population growth in terms of billions of people.

The world population growth rate is simply the difference between the birth rate and the death rate.

Example 11. Suppose on average there were 82 births per thousand people a year and 64 deaths. What would the population growth rate be?

Formula:

The world population growth rate is the difference between the birth rate and the death rate:

$$\text{growth rate} = \text{birth rate} - \text{death rate}$$

Carrying Capacity and Growth Models

For any particular species in a given environment, the **carrying capacity** is the maximum sustainable population. That is, it is the largest population the environment can support for extended periods of time.

Do you think the exponential growth of the human world population will continue forever?

Logistic Growth is a much better modal for populations than exponential growth.

Definiton 3. When the population is small relative to the carrying capacity, **logistic growth** is exponential with a fractional growth rate close to the base growth rate r. As the population approaches the carrying capacity, the logistic growth rate approaches zero. The fractional growth rate at any particular time depends on the population at that time as follows

$$\text{growth rate} = r \cdot \left(1 - \frac{\text{population}}{\text{carrying capacity}}\right)$$

Example 12. Assume that the Earth's carrying capacity is 12 billion people. Given that the population growth rate peaked in the 1960's at about 2.1%, when the population was about 3 billion, is it reasonable to assume that human population has been following a logistic growth pattern since the 1960s? Is it reasonable to assume that population has been growing logistically throughout the past century?

What do you think happens when a population **overshoots** is carrying capacity?

The following figure represents **collapse**

Lets read the Practical Matters section at the bottom of page 514.

Logarithmic Scales

Is a magnitude 7 earthquake really much stronger than a magnitude 4? Yes! Actually is it 33 thousand times stronger: The energy released in 33000 magnitude 4 earthquakes is equivalent to that of one magnitude 7 quake.

Consider table 8.4 on page 520.

Definiton 4. Earthquake Magnitude The magnitude scale for earthquakes is defined so that each magnitude represents about 32 times as much energy as the prior magnitude. More technically, the magnitude M is related to the released energy E by the following:

$$\log_{10} E = 4.4 + 1.5M \quad \text{or} \quad E = (2.5 \cdot 10^4) \cdot 10^{1.5M}$$

where the energy is measured in joules and magnitudes have no units.

Example 13. Compute the energy released in a magnitude 5.1, 5.3, and 5.7 earthquakes. How much energy was release in the magnitude 9.0 earthquake the triggered the tsunami? Compare with table 3.1 on page 157.

Sound

Sound is measured according to the Decibel scale, and common sound strengths are given on page 522 or the text.

Definiton 5. The loudness of a sound in decibels is defined by the following equivalent formulas:

$$\text{loudness in dB} = 10\log_{10}\left(\frac{\text{intensity of the sound}}{\text{intensity of softest audible sound}}\right)$$

$$\frac{\text{intensity of the sound}}{\text{intensity of softest audible sound}} = 10^{\frac{\text{loudness in dB}}{10}}$$

Example 14. How much more intense is a 75 decibel sound than a 14 decibel sound?

Sound follows the inverse square law which is stated as follows:

Definiton 6. The intensity of sound decreases with the square of the distance from the source, meaning that the intensity is proportional to $1/d^2$. We therefore say that sound following the **inverse square law** with distance.

Example 15. How does the sound of a siren at 30 meters compare the the same sound at 50 meters?

PH Scale of Acidity

Many household products as well as substances you would find in a chemistry lab have a **pH** quantity stated on them. This quantity is used to classify substances as **neutral, acidic, of basic**.

Definiton 7. Pure water is **neutral** and has a pH of 7.

Acids have a pH lower than 7.

Bases have a pH greater than 7.

Acidity is related to the concentration of positively charged hydrogen ions (hydrogen atoms without an electron) within a substance. We denote the concentration of such ions by $[\mathrm{H}^+]$ which is expressed in **moles** per liter. Where 1 mole is about $6 \cdot 10^{23}$ particles.

Definiton 8. The pH scale is defined by the following equivalent formulas:

$$\mathrm{pH} = -\log_{10}[\mathrm{H}^+] \quad \text{or} \quad [\mathrm{H}^+] = 10^{-pH}$$

where $[\mathrm{H}^+]$ is the hydrogen ion concentration in moles per liter.

Example 16. What is the pH of a solution with hydrogen ion concentration of 10^{-15} moles per liter? $5.3 \cdot 10^{-2}$? 3.2?

Acid Rain

Normal rain has a pH value of just under 6. Fossil fuel emissions significantly lower this value resulting in **acid rain**. Acid rain over Los Angeles has been observed to have a pH value of 2.0, equivalent to that of lemon juice. Such rain kills trees and plants, and waterlife.

Example 17. How much more acid is in rain of a pH value of 2 than that of 6?