

# Chapter 6 Notes

## The Definite Integral

### 1 Area

#### 1.1 What is Area?

**Definiton 1.** A **region** is a set of points in a plane and a **polygonal** region is a polygon, together with its interior.

We can always find the area of a polygon by cutting it up into triangles and finding the sum of the areas of the individual triangles.

**Definiton 2.** Let  $R$  be a region in the plane. We say that  $R$  is bounded if there is a polygonal region that entirely contains  $R$ .

**Definiton 3.** Let  $R$  be a region in the plane. If  $P$  is a polygonal region that contains  $R$ , we say  $P$  is a **circumscribed** polygon for  $R$ . If  $P$  is a polygonal region that lies entirely inside the region  $R$ , we call  $P$  an **inscribed polygon** of  $R$ .

**Definiton 4.** We call the greatest lower bound of areas of circumscribed polygons of  $R$  the **outer area** of  $R$ .

**Definiton 5.** We call the least upper bound of the areas of inscribed polygons of  $R$  the **inner area** of  $R$ .

**Definiton 6.** If the inner and outer area's of a region  $R$  are equal, we call the like value the area of  $R$ .

## 1.2 The Region under a Graph

The set of points in the plane given by

$$R = \{(x, y) | a \leq x \leq b, 0 \leq y \leq f(x)\}$$

is the **region** under  $f$  over  $[a, b]$

**Example 1.** We will compute the area of the region under the graph of  $f(x) = x^2$  over the interval  $[0, 2]$ . We first consider the outer area of this object. Consider the rectangle whose base is  $[0, 2]$  and whose height is 4. This rectangle is a circumscribed polygon for  $R$ . The area of the region under  $f$  must be less than this value.

Now let  $n \in \mathbb{N}$  and divide  $[0, 2]$  into  $n$  equal subintervals of length  $2/n$ . The  $k$ -th subinterval is given by  $[(2k-2)/n, 2k/n]$ . Now form  $n$  rectangles whose union is a circumscribed polygon for  $R$  whose height is  $f(2/n) = 4/n^2$ . The area of the circumscribed polygon is simply the sum of the areas of each rectangle. Since the height of each rectangle is determined by its right end-point, we call this sum the  $n$ -th **right-handed** sum, denoted  $R_f(n)$ .

We find

$$\begin{aligned}
R_f(n) &= f\left(\frac{2}{n}\right)\left(\frac{2}{n} - 0\right) + f\left(\frac{4}{n}\right)\left(\frac{4}{n} - \frac{2}{n}\right) + \cdots + f\left(\frac{2n}{n}\right)\left(2 - \frac{2n-2}{n}\right) \\
&= \frac{4}{n^2} \frac{2}{n} + \frac{16}{n^2} \frac{2}{n} + \cdots + \frac{4n^2}{n^2} \frac{2}{n} \\
&= \frac{4}{n^2} (1 + 4 + 9 + \cdots + n^2) \frac{2}{n} \\
&= \frac{8}{n^3} (1^2 + 2^2 + \cdots + n^2) \\
&= \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{4}{3} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right)
\end{aligned}$$

Taking the limit as  $n \rightarrow \infty$  we find  $R_f(n) \rightarrow 8/3$ . Thus the outer area can be no greater than  $8/3$ .

We now concentrate on finding the inner area of  $R$ . We divide  $[0, 2]$  into  $n$  subintervals as above. Thus time we construct rectangles contained in  $R$ . We define their sum to be the  $n$ -th **left handed** sum since their values are determined by the left endpoints. We calculate

$$\begin{aligned}
L_f(n) &= f(0)\left(\frac{2}{n} - 0\right) + f\left(\frac{2}{n}\right)\left(\frac{4}{n} - \frac{2}{n}\right) + \cdots + f\left(\frac{2n-2}{n}\right)\left(2 - \frac{2n-2}{n}\right) \\
&= 0 \frac{2}{n} + \frac{4}{n^2} \frac{2}{n} + \cdots + \frac{4(n-1)^2}{n^2} \frac{2}{n} \\
&= \frac{4}{n^2} (0 + 1 + 4 + \cdots + (n-1)^2) \frac{2}{n} \\
&= \frac{8}{n^3} (1^2 + 2^2 + \cdots + (n-1)^2) \\
&= \frac{8}{n^3} \frac{(n-1)n(2n-1)}{6} = \frac{4}{3} \left(1 - \frac{1}{n}\right) \left(2 - \frac{1}{n}\right)
\end{aligned}$$

As  $n \rightarrow \infty$   $L_f(n) \rightarrow 8/3$ . Thus the inner area of  $R$  cannot be less than  $8/3$ .

Was does this tell us the area under  $R$  must be?

### 1.3 Area as the Limit of a Sum

We can use the above approach for any function that is continuous and non-negative. For a function  $f(x)$  on an interval  $[a, b]$ , the algorithm is thus:

1. Subdivide the interval  $[a, b]$  into  $n$  pieces of equal length and let

$$\Delta x = \frac{b - a}{n}$$

Then define

$$x_0 = a \quad x_1 = a + \Delta x \quad x_2 = a + 2\Delta x$$

$$x_k = a + k\Delta x \quad x_n = a + n\Delta x = b$$

The set  $\{a = x_0, x_1, \dots, x_{n-1}, x_n = b\}$  forms a **subdivision** of  $[a, b]$ . Now let  $m_k$  be the point in the  $k$ -th subinterval  $[x_{k-1}, x_k]$  where  $f$  has its minimum, and let  $M_k$  be the point in the same subinterval where  $f$  has a maximum. The rectangle with base  $[x_{k-1}, x_k]$  and height  $f(m_k)$  is called a **lower rectangle** and the rectangle with the same base but height  $f(M_k)$  is called the **upper rectangle** for  $f$ .

The union of all lower rectangles is an inscribed polygon for the region  $R$  and the union of all the upper rectangles is an inscribed polygon for the region  $R$ .

The area of the inscribed polygon is called the  $n$ -th lower sum denoted  $LS_f(n)$  and is given by

$$\begin{aligned}
LS_f(n) &= f(m_1)\Delta x + f(m_2)\Delta x + \cdots + f(m_n)\Delta x \\
&= [f(m_1) + f(m_2) + \cdots + f(m_n)]\Delta x
\end{aligned}$$

The area of the circumscribed polygon is called the  $n$ -th upper sum denoted  $US_f(n)$  and is given by

$$\begin{aligned}
US_f(n) &= f(M_1)\Delta x + f(M_2)\Delta x + \cdots + f(M_n)\Delta x \\
&= [f(M_1) + f(M_2) + \cdots + f(M_n)]\Delta x
\end{aligned}$$

**Definiton 7.** If  $t_k$  is any point in the  $k$ -th subinterval, then the sum

$$S_f(n) = [f(t_1) + f(t_2) + \cdots + f(t_n)]\Delta x$$

is called the **Riemann sum** for  $f$ .

Thus we find the inequality

$$LS_f(n) \leq S_f(n) \leq US_f(n)$$

The area under the curve  $f$  is given by

$$\lim_{n \rightarrow \infty} S_f(n) = \lim_{n \rightarrow \infty} [f(t_1) + \cdots + f(t_n)] \frac{b-a}{n}$$

## 1.4 Monotone Functions

Let  $f$  be an increasing monotone function. Then we find

$$\begin{aligned}
R_f(n) - L_f(n) &= [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x - [f(x_0) + f(x_1) + \cdots + f(x_{n-1})]\Delta x \\
&= [f(x_n) - f(x_0)]\Delta x = [f(b) - f(a)]\frac{b-a}{n}
\end{aligned}$$

Letting  $n \rightarrow \infty$  we find  $R_f(n) = L_f(n)$ . Thus the region below any monotonic function over a closed interval has an area, and can be expressed as the limit of any associated Riemann sum.

## 2 Total Change

We found before that we can find a velocity function from a position function by taking its derivative. Now we will see that we can invert this process via anti-differentiation.

### 2.1 Recovering Distance from Velocity

Suppose that we fire a rocket into the air and compile the following list of data

$t$ (s)	0	0.5	1.0	1.5	2.0	2.5
$v$ (ft/s)	0	24	38	56	64	68

We will try to find the altitude of the rocket at various times. Let's look at what happened during the first half second of flight. Clearly, the velocity of the rocket will be constantly increasing during this time interval. If it was going 24 ft/s the entire time, its distance traveled would be  $24 \cdot .5 = 12$  ft. Moreover it could not have gone less than  $0 \cdot 0 = 0$  ft.

Now look at the second time interval. The maximum velocity of the rocket in this interval is 38 ft. So the rocket could not have traveled more than  $38 \cdot .5 = 19$  ft, or less than  $24 \cdot .5 = 12$  ft. Continuing this way we have the following table:

Time Interval	Min. Distance (ft)	Max. Distance (ft)
[0, 0.5]	0	12
[0.5, 1.0]	12	19
[1.0, 1.5]	19	28
[1.5, 2.0]	28	32
[2.0, 2.5]	32	34
Total	91	125

Thus we see that the altitude of the rocket after 2.5 seconds must be somewhere between 91 and 125 ft.

Class Question: How do we make this estimate better?

## 2.2 Recovering Distance from a Velocity Function

Now suppose we are given a non-negative velocity function  $v(t)$  instead of a set of data points. We want to determine how far an object has traveled on an interval  $[a, b]$ . We will do it in a similar way as the above.

Choose  $n \in \mathbb{N}$  and divide  $[a, b]$  into  $n$  time-intervals of length  $\Delta t = \frac{b-a}{n}$  and label the partition  $\{a = t_0, t_1, t_2, \dots, t_n = b\}$ . During the  $k$ -th subinterval, the initial velocity was  $v(t_{k-1})$  and the final velocity was  $v(t_k)$ .

If the velocity stayed at  $v_{k-1}$  on this interval, then the distance traveled would

be  $v(t_{k-1})\Delta t$ . If the velocity stayed at  $v_k$  on this interval, then the distance traveled would be  $v(t_k)\Delta t$ .

Summing all the distances, we find

$$L_v(n) = [v(t_0) + v(t_1) + \cdots + v(t_{n-1})]\Delta t$$

$$R_v(n) = [v(t_1) + v(t_2) + \cdots + v(t_n)]\Delta t$$

Letting  $\tau_k \in [t_{k-1}, t_k]$  we find the Riemann sum is given by

$$S_v(n) = [v(\tau_1) + v(\tau_2) + \cdots + v(\tau_n)]\Delta t$$

Thus if the velocity function is continuous or monotonic, the distance traveled can be found by taking the limit of the Riemann sum as  $n \rightarrow \infty$ .

### 2.3 Summation Notation

Summation notation is a very useful and cool way to write down the sum of several quantities. Consider the following

$$\sum_{i=m}^n f(k)$$

The symbol  $\Sigma$  is called the summation sign. The letter  $i$  is the dummy index. (You do not have to use  $i$ , you could use  $\xi$  if you wanted, hence the name dummy index).  $m$  and  $n$  are called the lower and upper limits of the sum and  $g(i)$  is called the summand. The following will illustrate the notation:

$$\sum_{k=1}^n k = 1 + 2 + 3 + \cdots + n$$

$$\sum_{i=1}^m \sqrt{i} =$$

$$\sum_{\xi=4}^{211} \xi^\pi =$$

We can now rewrite some of previously discussed sums in this notation:

$$L_f(n) =$$

$$R_r(n) =$$

$$LS_f(n) =$$

$$RS_f(n) =$$

$$S_f(n) =$$

Note that the sum is linear

$$\sum_{k=m}^n [f(k) + g(k)] = \sum_{k=m}^n f(k) + \sum_{k=m}^n g(k)$$

Other properties are summarized on page 367.

## 3 The Definite Integral

### 3.1 Generalized Riemann Sums

We now will generalize the Riemann sum method given in the last section.

Suppose we are given a function  $f$  over an interval  $[a, b]$ . Take the subdivision of  $[a, b]$  to be any collection of points

$$\sigma = \{x_0, x_1, \dots, x_n\}$$

such that

$$a = x_0 < x_1 < \dots < x_n = b$$

and set  $\Delta x_k = x_k - x_{k-1}$ . If all the  $\Delta_k$  have the same length, then we call the subdivision a **regular partition**. Now let  $t_k$  be any number in the  $k$ -th subinterval of  $\sigma$ ,  $x_{k-1} < t_k < x_k$  and consider the sum

$$S_f(\sigma) = \sum_{k=1}^n f(t_k) \Delta x_k$$

which we call the Riemann sum for  $f$  over  $[a, b]$  relative to  $\sigma$ . Since  $f$  is bounded over each subinterval, let  $B_k$  be the least upper bound of  $f$  over  $[x_{k-1}, x_k]$  and let  $b_k$  be the greatest lower bound of  $f$  over the same interval. We call

$$US_f(\sigma) = \sum_{k=1}^n B_k \Delta x_k$$

the upper sum for  $f$  over  $[a, b]$  relative to  $\sigma$ , and

$$LS_f(\sigma) = \sum_{k=1}^n b_k \Delta x_k$$

the lower sum of  $f$  over  $[a, b]$  relative to  $\sigma$ . Note the  $LS_f(\sigma) \leq US_f(\sigma)$ .

If  $\sigma_1$  and  $\sigma_2$  are any two subdivisions of  $[a, b]$  such that every point of  $\sigma_1$  is also a point of  $\sigma_2$ , we call  $\sigma_2$  a **refinement** of  $\sigma_1$ . Note  $\sigma_1 \cup \sigma_2$  is a refinement of both  $\sigma_1$  and  $\sigma_2$ . Moreover, we have the inequality

$$LS_f(\sigma_1) \leq LS_f(\sigma_1 \cup \sigma_2) \leq US_f(\sigma_1 \cup \sigma_2) \leq US_f(\sigma_2)$$

**Theorem 8.** *If a function  $f$  is continuous on the interval  $[a, b]$ , then there is a unique number  $I$  such that for every subdivision  $\sigma$  of  $[a, b]$*

$$LS_f(\sigma) \leq I \leq US_f(\sigma)$$

**Proof:**

Exercise.

□

### 3.2 Integrable Functions

**Definiton 9.** If a function  $f$  is bounded on  $[a, b]$  and there is a unique number  $I$  such that for every subdivision  $\sigma$  of  $[a, b]$

$$LS_f(\sigma) \leq I \leq US_f(\sigma)$$

we say  $f$  is integrable on  $[a, b]$  and we call the number  $I$  the definite integral of  $f$  over  $[a, b]$ , denoted by

$$I = \int_a^b f(x)dx$$

**Theorem 10.** *If a function  $f$  is continuous on  $[a, b]$ , then  $f$  is integrable on  $[a, b]$ .*

**Theorem 11.** *If a function  $f$  is monotonic on  $[a, b]$ , then  $f$  is integrable on  $[a, b]$ .*

### 3.3 The Limit of Riemann Sums

If a function  $f$  is defined on  $[a, b]$ , every Riemann sum for  $f$  over  $[a, b]$  has the form

$$S_f(\sigma) = \sum_{k=1}^n f(t_k) \Delta x_k$$

for some subdivision  $\sigma$  with each  $t_k$  in the  $k$ -th subinterval of  $\sigma$ . We say the Riemann sums for  $f$  over  $[a, b]$  have a limit  $L$  if for every  $\epsilon > 0$ , there is a subdivision  $\sigma$  of  $[a, b]$  such that  $|S_f(\sigma) - L| < \epsilon$  for all possible choices of  $t_k$ .

We use the notation

$$\lim_{\sigma} \sum_{k=1}^n f(t_k) \Delta x_k = L$$

to mean that the Riemann sums for  $f$  have limit  $L$ .

**Theorem 12.** *If a function  $f$  is integrable on  $[a, b]$  then the Riemann sums for  $f$  over  $[a, b]$  have limit  $\int_a^b f(x) dx$ , which is to say*

$$\int_a^b f(x) dx = \lim_{\sigma} \sum_{k=1}^n f(t_k) \Delta x_k$$

### 3.4 Notation for the Definite Integral

For an definite integral given by

$$\int_a^b f(t)dt$$

the symbol  $\int$  is called the integral sign. Then number  $a$  is called the lower limit. The number  $b$  is called the upper limit. The function  $f(t)$  is called the integrand. The variable  $t$  is called the dummy variable since

$$\int_a^b f(t)dt = \int_a^b f(\eta)d\eta$$

### 3.5 Interpretations of the Definite Integral

**Theorem 13.** *If  $f$  is integrable on  $[a, b]$ , then*

- 1. If  $f$  is non-negative over  $[a, b]$ , then the region below  $f$  over  $[a, b]$  has area equal to the definite integral.*
- 2. If  $f$  is negative over  $[a, b]$ , then the region above  $f$  and below  $[a, b]$  has area and is equal to the negative of the definite integral.*
- 3. If  $f$  is partly positive and partly negative, then the regions between  $f$  and the interval  $[a, b]$  have area and the definite integral is the difference between the areas.*

### 3.6 Evaluating the Definite Integral

If a function  $f$  is integrable on  $[a, b]$ , then we may use any Riemann sum to approach the definite integral. We will use the formula

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} R_f(n) = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \frac{b-a}{n}$$

where  $x_k = a + k \cdot \frac{b-a}{n}$ .

**Example 2.** Evaluate  $\int_{-1}^2 x^3 dx$

The bad thing about this method is that it takes a long time. There are sometimes a few tricks one can use to evaluate integrals; however the process is difficult in general until we arrive to the fundamental theorem of calculus.

**Example 3.** Evaluate  $\int_{-1}^1 \sqrt{1-x^2} dx$

**Example 4.** Using the fact that  $\sin x$  is an odd function, evaluate  $\int_{-a}^a \sin x$ .

**Example 5.** Using the fact that  $\cos x$  is an even function simplify  $\int_{-b}^b \cos x$

## 4 Estimating Definite Integrals

Often times, one cannot compute an exact answer for a definite integral.

However, one can always approximate a solution. This section is a summary of such methods.

### 4.1 Left-Hand and Right-Hand Sums

Given a function  $f$  defined on  $[a, b]$ , we can choose a positive integer  $n$  and

define a regular partition  $\sigma$  of  $[a, b]$  with  $n$  intervals each with length

$\Delta x = \frac{b-a}{n}$ . Setting  $x_k = a + k\Delta x$  we previously found

$$L_f(n) = \sum_{k=0}^{n-1} f(x_k)\Delta x$$
$$R_f(n) = \sum_{k=1}^n f(x_k)\Delta x$$

If  $f$  is integrable, both of these sums approach the value of the definite integral. Thus we see

$$\int_a^b f(x)dx \approx L_f(n) \approx R_f(n)$$

## 4.2 Behavior of Left and Right sums

Consider the integral

$$\int_0^2 x^2 dx = \frac{8}{3}$$

We will see how the left and right hand sums behave for this integral for various  $n$ .

$n$	$L_f(n)$	$L_f$ error	$R_f(n)$	$R_f$ error
5	1.92	-0.746667	3.52	0.853333
10	2.28	-0.386667	3.08	0.413333
20	2.47	-0.196667	2.87	0.203333
40	2.5675	-0.099167	2.7675	0.100833
80	2.616875	-0.049792	2.2716875	0.050208
160	2.64172	-0.024948	2.691719	0.025052

Since  $f$  is increasing on the domain of integration, the left hand sum is an underestimate and the right hand sum is an overestimate. Note each time we multiply  $n$  by 2, the error decreases by a factor of two. This is known as **linear convergence**.

**Theorem 14.** *If a function  $f$  is differentiable and  $|f'(x)| \leq B$  for all  $x \in [a, b]$ , then the error in estimating the definite integral  $\int_a^b f(x)dx$  by  $L_f(n)$  or  $R_f(n)$  is no greater than*

$$\frac{B(b-a)^2}{2n}$$

**Example 6.** How many subintervals would be necessary to give a 7-decimal approximation of  $\int_0^2 x^2 dx$  using  $L_f(n)$  or  $R_f(n)$ ?

Note that this method takes a lot of time. There is a better one.

### 4.3 Trapezoidal Estimate

You might think that if we average the left and right sums that we would get rid of a lot of error. Define a new sum  $T_f(n)$  by the equation

$$T_f(n) = \frac{1}{2}[L_f(n) + R_f(n)]$$

We call this the trapezoidal sum as illustrated in the following:

The following table demonstrates the trapezoidal error for the integral  $\int_0^2 x^2 dx$ .

$n$	$T_f(n)$	$T_f$ error
5	2.72	0.053333
10	2.68	0.013333
20	2.67	0.003333
40	2.6675	0.000833
80	2.666875	0.000208
160	2.666719	0.000052

One can see that when we double  $n$ , the error decreases by a factor of 4. This is called **quadratic** convergence.

**Theorem 15.** *If a function  $f$  is twice differentiable and  $|f''(x)| \leq B$  for  $x \in [a, b]$ , then the error in estimating the definite integral  $\int_a^b f(x)dx$  by  $T_f(n)$  is no greater than*

$$\frac{B(b-a)^3}{12n^2}$$

**Example 7.** How large of an  $n$  is needed to approximate the above integral to seven decimal places?

## 4.4 The Midpoint Sum

The midpoint sum approximation is roughly as good as the trapezoid approximation. We demonstrate it now. Let  $\mu_k$  be the midpoint of the  $k$ -th subinterval given explicitly by

$$\mu_k = \frac{1}{2}(x_{k-1} + x_k)$$

The midpoint sum is

$$M_f(n) = [f(\mu_1) + f(\mu_2) + \cdots + f(\mu_n)]\Delta x$$

The following is a summary of an application of the midpoint approximation to the integral  $\int_0^2 x^2$ :

$n$	$M_f(n)$	$M_f$ error
5	2.64	-0.026667
10	2.66	-0.006667
20	2.665	-0.001667
40	2.66625	-0.000417
80	2.666563	-0.000104
160	2.666641	-0.000026

The negative errors indicate this approximation underestimates the true value of the integral. The approximation also exhibits quadratic convergence.

**Theorem 16.** *If a function  $f$  is twice differentiable and  $|f''(x)| \leq B$  for  $x \in [a, b]$ , then the error in estimating the definite integral  $\int_a^b f(x)dx$  by  $M_f(n)$*

is no greater than

$$\frac{B(b-a)^3}{24n^2}$$

Thus we see this rule is slightly better than the trapezoid rule.

## 4.5 Simpson's Rule

This is the fastest converging approximation method we will consider.

Combining the trapezoid and midpoint sums, we define a new sum by

$$SR_f(2n) = \frac{1}{3}[2 \cdot M_f(n) + T_f(n)]$$

Computing  $SR_f(10)$  for the previous integral yields the exact answer, so we will consider another example. We will later find that

$$\int_1^2 \frac{1}{x} dx = \ln 2 \approx 0.69314718$$

We make the error table:

$n$	$T_f(n)$	$T_f$ error
4	0.69325397	0.00010679
8	0.69314765	0.00000735
16	0.69314765	0.00000047
32	0.69314721	0.00000003

Each time we double  $n$ , we decrease the error by a factor of 16. This is very good.

**Theorem 17.** *If a function  $f$  is four times differentiable and  $|f^{(4)}(x)| \leq B$  for all  $x \in [a, b]$ , then the error in estimating the definite integral  $\int_a^b f(x)dx$  by  $SR_f(n)$  is no greater than*

$$\frac{B(b-a)^5}{180n^4}$$

From this we see that for a polynomial of degree 3, Simpson's rule will be exact.

## 5 Properties of Definite Integrals

In this section we summarize a few properties of definite integrals.

### 5.1 Linearity Properties

**Theorem 18.** *If a function  $f$  is integrable on  $[a, b]$  and  $c$  is a constant, then the function  $cf$  is integrable on  $[a, b]$  and*

$$\int_a^b cf(x)dx = c \int_a^b f(x)dx$$

**Proof:**

The proof follows from the fact that

$$\int_a^b cf(x)dx = \lim_{\sigma} \sum_{k=1}^n cf(t_k)\Delta x_k$$

□

**Theorem 19.** *If functions  $f$  and  $g$  are integrable on  $[a, b]$ , the  $f + g$  is integrable on  $[a, b]$  and*

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

## 5.2 Interval Properties

**Definiton 20.** *If a function  $f$  is defined at  $a$  then*

$$\int_a^a f(x)dx = 0$$

*If  $a < b$  and  $\int_a^b f(x)dx$  exists, then*

$$\int_b^a f(x)dx = - \int_a^b f(x)dx$$

**Theorem 21.** *If  $a < b < c$  and a function  $f$  is integrable on  $[a, b]$  and  $[b, c]$ , then  $f$  is integrable on  $[a, c]$  and*

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

**Theorem 22.** *If a function  $f$  is integrable on an interval  $I$ , then  $f$  is integrable on any subinterval of  $I$ . If  $a, b$  and  $c$  are any points in  $I$ , then*

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

*without regard for the order in which  $a, b$  and  $c$  occur.*

### 5.3 Order Properties

**Theorem 23.** *If  $a < b$ , a function  $f$  is defined and continuous over  $[a, b]$ , and  $f \geq 0$  everywhere, then  $\int_a^b f(x)dx \geq 0$ .*

**Theorem 24.** *If  $a < b$ , and two functions  $f$  and  $g$  are defined and continuous on  $[a, b]$ , with  $f(x) \leq g(x)$  for all  $x$ , then  $\int_a^b f(x)dx \leq \int_a^b g(x)dx$ .*

Now we will see how we can use these theorems.

**Example 8.** Find bounds on the definite integral

$$\int_2^3 \frac{1}{x}$$

**Theorem 25.** *If a function  $f$  is integrable on  $[a, b]$ , then  $|f|$  is integrable on  $[a, b]$  and*

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$$

### 5.4 Average Value of a Function

If a function  $f$  is integrable and positive over  $[a, b]$ , then  $\int_a^b f(x)dx$  is the area below  $f$ . If there is a rectangle with base  $[a, b]$  that has the same area as this region, we will call the height  $h$  of the rectangle the average value of  $f$ .

Equating areas we find

$$h(b - a) = \int_a^b f(x)dx \rightarrow h = \frac{1}{b - a} \int_a^b f(x)dx$$

**Theorem 26.** (*The Mean Value Theorem for Integrals*). If a function  $f$  is defined and continuous on  $[a, b]$ , then there is a point  $z \in [a, b]$  such that

$$f(z) = \frac{1}{b-a} \int_a^b f(x) dx$$

**Theorem 27.** If two functions  $f$  and  $g$  are defined and continuous and  $g$  is non-negative over the interval  $[a, b]$ , then there is a point  $z \in [a, b]$  such that

$$\int_a^b f(x)g(x)dx = f(z) \int_a^b g(x)dx$$

## 6 The Fundamental Theorem of Calculus

**Definiton 28.** If  $f$  is continuous over the interval  $[a, b]$ , the function

$$A(x) = \int_a^x f(t)dt$$

is called the **area function** for  $f$ , as can be seen by the following picture.

We compute via definition of the derivative

$$\frac{dA}{dt} = \lim_{h \rightarrow 0} \frac{A(x+h) - A(x)}{h}$$

One can show as the book does on page 405, via the mean value theorem of integrals, that:

**Theorem 29.** If a function  $f$  is continuous on  $[a, b]$  and

$$A(x) = \int_a^x f(t)dt$$

then  $A$  is differentiable in  $(a, b)$  and is an antiderivative of  $f$ . That is,

$A'(x) = f(x)$  for every  $x \in (a, b)$ .

## 6.1 Definite Integrals by Antiderivatives

Note by the previous theorem we immediately find that

$$A(b) = \int_a^b f(t)dt = A(b)$$

Now suppose that  $g$  is any antiderivative of  $f$ . Since  $A$  is also an antiderivative of  $f$ , we know that  $A$  and  $g$  differ only by a constant,

$$A(x) = g(x) + C$$

Since  $A(a) = 0$ , we find

$$0 = A(a) = g(a) + C$$

which gives  $C = -g(a)$  which implies  $A(x) = g(x) - g(a)$ . Evaluating at  $x = b$  we find

$$\int_a^b f(t)dt = A(b) = g(b) - g(a)$$

which is the fundamental theorem of calculus. It says, given a function  $f$ , to find the integral of  $f$  over  $[a, b]$  we need to find its antiderivative  $g$ , and then

evaluate  $g$  at  $a$  and  $b$  and take the difference of these values. We now state this in theorem form

**Theorem 30.** *If a function  $f$  is continuous on  $[a, b]$  and  $g$  is any antiderivative of  $f$  on  $[a, b]$ , then*

$$\int_a^b f(x)dx = g(b) - g(a)$$

**Example 9.** Apply the fundamental theorem of calculus to evaluate the following integrals.

$$\int_2^4 x^3 dx \quad \int_{-\pi/4}^{\pi/4} \sec^2 x dx \quad \int \frac{1}{x\sqrt{x^2-1}} dx$$

*This process is called **integration**.*

## 7 Integration by Substitution

In general integration is a hard thing to do. We will now see the simplest method for evaluating non-trivial integrals.

Sometimes, we can transform a relatively complicated integral into an easier one. The following examples will demonstrate this idea.

**Example 10.** Find

$$\int (x+3)^7 dx$$

**Example 11.** Find

$$\int \sin ax dx$$

**Example 12.** Find

$$\int x\sqrt{x^2 + a^2} dx$$

**Example 13.** Find

$$\int_0^3 (3x - 2)^{1/2} dx$$

**Example 14.** Find

$$\int_0^{\pi/4} \sin^2 x \cos x dx$$

**Example 15.** Find

$$\int x\sqrt{2x + 5} dx$$