Calculus Solutions: Chapter 1.1

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1. Under what conditions is the union of two intervals another interval? Does your answer depend on the type of interval (open, closed, infinite)?

Solution:
The union of any two intervals is another interval if the two intervals are not disjoint and contain at least one common element. For a more explicit argument, consider the two open intervals

$$(a,b) = \{x \in \mathbb{R} | a < x < b\} \quad (c,d) = \{x \in \mathbb{R} | a < x < b\}$$

and suppose each contain a common point $e \in \mathbb{R}$. Then

$$\begin{gathered}(a,b) \cup (c,d) = \{x \in \mathbb{R} | a < x \leq e\} \cup \{x \in \mathbb{R} | e < x \leq b\} \\ \cup \{x \in \mathbb{R} | c < x \leq e\} \cup \{x \in \mathbb{R} | e \leq x < d\} \\ = \{x \in \mathbb{R} | a < x \leq e\} \cup \{x \in \mathbb{R} | e < x < d\} = \{x \in \mathbb{R} | a < x < d\} = (a,d)\end{gathered}$$

The answer does not depend on the type of interval, and parallel proofs of the above may be created for combinations of open, closed, and infinite intervals.

□

Use the properties of the real number system given in this section to prove the following; you may also use each result to prove subsequent results, if appropriate.

4a.(ii) For $a, b, c \in \mathbb{R}$, assume that if $a = b$, then $a + c = b + c$ and $ab = bc$. Prove the cancelation law: if $ac = bc$ and $c \neq 0$, then $a = b$.

Solution:
We first note that since $c \neq 0$, by property 5 on page 4 of the text, c has a multiplicative inverse $1/c$ such that $c \cdot (1/c) = 1$. Thus we multiply both sides of the above equation by the inverse element $1/c$ and obtain

$$\frac{1}{c}(ac) = \frac{1}{c}(bc)$$

Applying property 1d on page 4 we find

$$(ac)\frac{1}{c} = (bc)\frac{1}{c}$$

Applying property 1b we have

$$a\left(c \cdot \frac{1}{c}\right) = b\left(c \cdot \frac{1}{c}\right)$$

and finally noting that by definition of the multiplicative inverse of c, $c \cdot \frac{1}{c} = 1$, we see

$$a = b$$

$\square$

4c. Show for every $a \in \mathbb{R}$ that $a \cdot 0 = 0$.

Solution:
From property 2, we note that the set $\mathbb{R}$ has an additive identity element denoted 0 such that if $a \in \mathbb{R}$ then

$$a + 0 = a$$

Multiplying both sides of this equation by a we find

$$a \cdot a + a \cdot 0 = a \cdot a$$

By property 4, the element $a \cdot a \in \mathbb{R}$ has an additive inverse $-(a \cdot a)$ such that $a \cdot a - (a \cdot a) = 0$. Adding this additive inverse to both sides of the above equation yields

$$-(a \cdot a) + a \cdot a + a \cdot 0 = -(a \cdot a) + a \cdot a$$

$$a \cdot 0 = 0$$

$\square$

4e. Prove that $0 < 1$ by carrying out the following steps:
i. Show that if $0 = 1$, then for every $a \in \mathbb{R}$, $a = 0$. Conclude that $0 \neq 1$.
ii. Show that if $1 < 0$, then $0 < -1$ and hence $-1 < 0$. Conclude that $1 < 0$ is impossible.
iii. Conclude that $0 < 1$, and explain why.

Solution:
Suppose that the multiplicative and additive identity elements of $\mathbb{R}$ are equal, $(1 = 0)$. Let $a \in \mathbb{R}$ and consider

$$a \cdot 1 = a \cdot 0$$

By Property 3 on page 4 and exercise 4c the above equation becomes

$$a = 0$$

Thus $a = 0$ for any $a \in \mathbb{R}$. Since, $a = 3 \in \mathbb{R}$ this is clearly a contradiction. Therefore $1 \neq 0$.

Consider the following lemma

Lemma 1. *If* $\{a, r, t\} \in \mathbb{R}$ *and* $a < 0, r < t$, *then* $ar > at$.

Proof:
Let $a = -A$ and note $A > 0$. Then since $t > r$ we find

$$t - r > 0 \Rightarrow -(r - t) > 0$$

Applying property 6d on page 4 we have

$$-A(r - t) > 0 \cdot A = 0$$

which be definition of A yields

$$a(r - t) > 0 \Rightarrow ar > at$$

□

Now suppose $1 < 0$. By the above lemma this implies $-1 \cdot 1 = -1 > -1 \cdot 0 = 0$. Thus $0 < -1$. Adding this result to our assumption, we find $1 < -1$. Thus by exercise 4d(ii) there exists a $c > 0$ such that $1 + c = -1$. Solving we find $c = -2 \not> 0$. Thus we have a contradiction and we may conclude $1 \not< 0$.

Thus sine $1 \neq 0$ and $1 \not< 0$ we conclude by property 6a on page 4 that $0 < 1$.

□

Give the interval in which x lies if

7b. $|x - 5| < 1.5$

Solution:
Rewriting the inequality, we have

$$\pm(x - 5) < 1.5$$

This yields the two equations

$$x - 5 < 1.5 \quad \text{and} \quad x - 5 > -1.5$$

Solving we find $x < 6.5$ and $x > 3.5$, or $3.5 < x < 6.5$. Putting this in interval notation $x \in (3.5, 6.5)$.

$\square$

7d. $|x+4| < 0.001$
Using the same method as in the previous problem we find
$-4.001 < x < -3.999$ or in interval notation $x \in (-4.001, -3.999)$.

$\square$

9. Prove that if S has a greatest lower bound, it is unique.

Solution:
Let S have a greatest lower bound L, and suppose there exists another greatest lower bound L' of S such that $L' \neq L$. By property 6d on page 4, either $L < L'$, $L > L'$, or $L = L'$. Suppose $L < L'$. Then L is not a greatest lower bound, since L' is also a lower bound of S but is larger than L. Similarly if $L' < L$, then L' is not a greatest lower bound. Therefore the only other option is $L = L'$, which shows the uniqueness of the greatest lower bound.

$\square$

Give the least upper bounds and the greatest lower bound of the following subsets of $\mathbb{R}$. If a bound does not state it.

11b. $(0, 1)$

Solution:
The least upper bound of this set is 1. The greatest lower bound is 0.

$\square$

11d. $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$

Solution:
The least upper bound of this set is 1. The greatest lower bound is 0.

$\square$

11f. $\{x | x > 0 \text{ and } x^2 < 2\}$

Solution:
We note this set is a representation of the interval $(0, \sqrt{2})$. Thus the least upper bound is 0 and the greatest lower bound is $\sqrt{2}$.

$\square$

13. Let S be a nonempty, finite set of real numbers.

a) Explain why S is bounded.
b) What is the least upper bound of S?
c) What is the greatest lower bound of S?

Solution:
Let the total number of elements of S be n. Since the real numbers are well ordered, (for any two real numbers a and b either $a = b$, $a < b$, or $b > a$). we may represent S as follows

$$S = \{s_1, s_2, \dots, s_n\}$$

where the s_i are the n elements of S and they are ordered such that $s_i \geq s_{i+1}$ for all i. Then S must be bounded since $s_n \in \mathbb{R}$ and is greater than or equal s_i for all i. Moreover, s_n is the least upper bound of the set. Similarly, s_1 is the greater lower bound of S.

$\square$