On Smooth Surfaces of Degree ≥ 11 in the Projective Fourspace

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Introduction

Smooth varieties with small invariants have got renewed interest in recent years, primarily due to the finer study of the adjunction mapping by Reider, Sommese and Van de Ven [R], [So], [SV], [VdV]. For the special case of smooth surfaces in $\mathbb{P}^4$ the method goes back to the italian geometers, who at the turn of the century used it for the study of the surfaces of degree less than 7, or sectional genus $\pi \leq 3$. Later on, for bigger invariants, there are contributions by Commesatti and especially Roth. For example, in [Ro], Roth tried to establish a classification of smooth surfaces with $\pi \leq 6$, but his lists are incomplete since he misses the non-special rational surfaces of degree 9 and the minimal bielliptic surfaces of degree 10. Nowadays, through the effort of several mathematicians (see references below), a complete classification of smooth surfaces in $\mathbb{P}^4$ has been worked out up to degree 10.

But, apart the general framework of classification problems concerning codimension two varieties, there is also another strong motivation for the interest in these surfaces. Namely, in a recent paper Ellingsrud and Peskine [ElP1] proved the conjecture of Hartshorne that there are only finitely many families of special surfaces in $\mathbb{P}^4$. More specifically, given an integer $a < 6$, they show that the degree of smooth surfaces with $K^2 < a\chi$ is bounded. In particular, there are only finitely many families of smooth surfaces in $\mathbb{P}^4$, not of general type. However, the question of an exact degree bound is still open. A recent work of Braun and Floystad [BF] improves the initial bound (~ 10000) of Ellingsrud and Peskine to $d \leq 105$, but it is believed that the degree of the smooth, non-general type surfaces in $\mathbb{P}^4$ should be less or equal to 15. A similar finiteness result for 3-folds in $\mathbb{P}^5$ was recently proved in [BOSS], but the real degree bound is supposed to be much higher in this case. Nevertheless, examples of smooth 3-folds in $\mathbb{P}^5$ not of general type are known only up to degree 18.

Another reason for the interest in studying surfaces in $\mathbb{P}^4$ originates in the small number of known liaison classes of such surfaces. On the other side, Proposition 0.32 below also accounts for such a spareness behavior, so each new specimen of liaison classes is of real interest. In this direction, the work of Decker, Ein and Schreyer [DES] provides a powerful and effective method of construction of surfaces in $\mathbb{P}^4$. Their method, whose basic idea is the application of Beilinson's spectral sequence to construct the ideal sheaf $\mathcal{I}_S$ from the cohomology modules H^1 and H^2, will be recalled in detail in chapter 2 below.

The aim of this paper is to provide various examples of smooth surfaces in $\mathbb{P}^4$ of degree ≥ 10 and to describe their geometry. On one side we give an account of a rough attempt of classification of smooth surfaces of degree 11 in $\mathbb{P}^4$ and provide constructions for 22 different families of surfaces in this degree, being able in some cases to show also uniqueness. Among the described families, 10 are of surfaces not of general type. Three of these examples were previously described in [DES]. On another side we provide a series of examples of smooth surfaces in $\mathbb{P}^4$, not of general type, in degrees varying from 12 up to 15. We've tried to work out examples of higher degrees but we failed in this attempt. The methods of construction we used are mainly the Eagon-Northcott approach of [DES] and liaison techniques. The most remarkable families we found are:

- minimal proper elliptic surfaces of degree 12 and sectional genus $\pi = 13$,

- non-minimal $K3$ surfaces of degree 14 and sectional genus 19, and

- non-minimal abelian surfaces of degree 15, sectional genus 21, lying on only one quintic hypersurface and thus not coming via a $(5,5)$ liaison from the Horrocks-Mumford torus of degree 10. We remark here that the quintic elliptic scrolls, the bielliptic surfaces of degree 10 and 15, the minimal abelian surfaces of degree 10 and the non-minimal abelian surfaces of degree 15 arising of these via liaison are essentially the only other known smooth irregular surfaces in $\mathbb{P}^4$ (up to taking the pullback through a finite morphism $\mathbb{P}^4 \to \mathbb{P}^4$).

At this point it may be appropriate to recall some references for the list of the smooth surfaces in degrees less or equal to 10 in $\mathbb{P}^4$. The classification and construction of surfaces of degree ≤ 7 is initiated in [Ro] and completed up to degree 8 in [Io1], [Io2], [Ok2], [Ok3], [Ok4], supplemented for the case of rational surfaces of degree 8, sectional genus 5 by [Al1]. In degree 9, the rational surfaces are described in [Al1] and [Al2], the Enriques surfaces with $\pi = 6$ in [Cos] and [CV], while the classification and description of the liaison classes is completed in [AR]. In degree 10, the classification in terms of numerical invariants and the description of most of the surfaces is achieved in the beautiful thesis of K. Ranestad [Ra1]. The existence and the uniqueness of bielliptic surfaces of degree 10, $\pi = 6$ was taken care by [Ser], the Enriques surfaces of degree 10, $\pi = 8$ were first constructed in [DES] and further studied in [Br], while the minimal abelian surfaces were first described about 20 years ago in [HM]. We provide here two constructions of a non-minimal $K3$ surface of degree 10, $\pi = 9$, lying on only one quartic hypersurface, and thus give a positive answer to the last existence case in [Ra1]. The left uniqueness problems, the syzygies and the description of the liaison classes in degree 10 are completed in [PR].

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0. Notations and basic results

We will use standard notations as for instance those in [Ha], [BPV] or [GH]. The invariants of a smooth surface S in $\mathbb{P}^n$ are denoted as follows:

$d = d(S) = H^2$ the degree of S, where H is the hyperplane class
$\pi = \pi(S)$ the genus of a generic hyperplane section
$p_g = p_g(S) = h^0(\mathcal{O}_S(K)) = h^2(\mathcal{O}_S)$ the geometric genus of S, where K is the canonical class
$q = q(S) = h^1(\mathcal{O}_S)$ the irregularity of S
$\chi = \chi(S) = \chi(\mathcal{O}_S)$ the Euler characteristic of the surface
$s = h^1(\mathcal{O}_S(1)) = h^1(\mathcal{O}_S(H))$ the speciality of S, see [Al1]
$\kappa(S)$ the Kodaira dimension of S
$p_a(C)$ the arithmetic genus of a curve C on S
$g(C)$ the geometric genus of a smooth curve C on S.

In blowing-up situations we will denote in the same way a divisor downstairs and its total transform on the blow-up. Also a rational curve C with self-intersection $C^2 = -1$ will be called a (-1) curve and analogously we'll speak of a (-2) curve when $C^2 = -2$. A (-1) curve will be called a (-1) line if it has degree 1 with respect to the current very ample linear system on the surface.
For C a curve on S the adjunction gives a canonical divisor on the curve C:

$$K_C \equiv (C+K)|_C$$

hence the arithmetic genus of C can be computed by the following:

Adjunction formula 0.1. $2p_a(C) - 2 = C^2 + C \cdot K$.

Proof. See [Ha Prop. 1.5].□

For curves C, D and $C \cup D$ lying on a smooth surface S the formula yields the following addition rule for the arithmetic genus :

$$p_a(C \cup D) = p_a(C) + p_a(D) + C \cdot D - 1. \qquad (0.2.)$$

Theorem (Riemann-Roch) 0.3.

$$\chi(\mathcal{O}_S(C)) = \mathrm{h}^0(S, \mathcal{O}_S(C)) - \mathrm{h}^1(S, \mathcal{O}_S(C)) + \mathrm{h}^0(S, \mathcal{O}_S(K - C)) = \frac{1}{2}(C^2 - C \cdot K) + \chi(S).$$

Proof. See [Ha Th.1.6].□

Hodge index theorem 0.4. *If D is an effective divisor on S with $D^2 > 0$ and C is a divisor such that $D \cdot C = 0$, then $C^2 < 0$ or $C \equiv 0$.*

Proof. See [Ha Th.1.9].

The index theorem may be used to bound the self intersection number of a curve on S. Namely one has the following:

Corollary 0.5. *Let D be an effective divisor with $D^2 > 0$ on a smooth surface S. If C is a divisor on S then*

$$C^2 \leq \frac{(D \cdot C)^2}{D^2}.$$

Proof. Apply the index theorem to $C - (\frac{D \cdot C}{D^2})D$.□

For smooth surfaces in $\mathbb{P}^4$ with normal bundle N_S one obtains the equality:

$$d^2 - c_2(N_S) = d^2 - 10d - 5H \cdot K - 2K^2 + 12\chi(S) = 0, \tag{0.6.}$$

which expresses the fact that S has no double points. This will be in the sequel referred to as the double point formula.

Theorem (Severi) 0.7. *All smooth surfaces in $\mathbb{P}^4$, except for the Veronese surface, are linearly normal.*

Proof. See [Sev] or [Mo].□

(0.8.) Multisecants.(see [LB]) Some classical numerical formulas for multisecant lines to a smooth surface in $\mathbb{P}^4$ have been recently studied again by Le Barz. Consider the double curve Γ of a general projection of such a surface S to $\mathbb{P}^3$ and denote by

$$\delta = \binom{d-1}{2} - \pi$$

the degree of Γ, by

$$t = \binom{d-1}{3} - \pi(d-3) + 2\chi - 2$$

the number of apparent triple points, i.e. the number of trisecants to S which meet a general point and by

$$h = \frac{1}{2}(\delta(\delta - d + 2) - 3t)$$

the number of apparent double points on Γ.
The number of 4-secants to S which meet a general line (if finite) is :

$$N_4(d, \pi, \chi) = 2\binom{d}{4} + t(d-3) + h - \delta\binom{d-3}{2}. \tag{0.9.}$$

The number of 5-secants to S which meet a general plane (if finite) is:

$$\begin{aligned} N_5(d, \pi, \chi) =& \frac{1}{24}d(d-3)(d-4)(d^2 - 15d + 2) - \binom{\delta}{2}(d-4) \\ & - \frac{\delta}{6}(d-2)(d-4)(d-21) + h(d-8) + \delta t - 3t(d-3). \end{aligned} \tag{0.10.}$$

Suppose there are no lines on S with positive self-intersection. Then the number of 6-secants (if finite) plus the number of exceptional lines on S is:

$$
\begin{aligned}
(0.11.) \qquad N_6(d,\pi,\chi) = & -\frac{1}{144}d(d-4)(d-5)(d^3+30d^2-577d+786) \\
& + \delta(2\binom{d}{4} + 2\binom{d}{3} - 45\binom{d}{2} + 148d - 317) \\
& - \frac{1}{2}\binom{\delta}{2}(d^2-27d+120) - 2\binom{\delta}{3} \\
& + h(\delta - 8d + 56) + t(9d - 3\delta - 28) + \binom{t}{2}.
\end{aligned}
$$

(0.12.) Adjunction theory. In order to determine the possible numerical invariants of the various smooth surfaces in $\mathbb{P}^4$ and to see how the constructed surfaces fit into the Enriques-Kodaira classification we will use adjunction theory [So], [VdV], [SV]:

Theorem (Adjunction mapping) 0.13. *Let S be a smooth surface, H be a very ample divisor and K the canonical divisor. Then $\mid H+K \mid$ is non-special and has dimension $N = \pi + \chi - 2$. Furthermore*
(I) $\mid H+K \mid = \emptyset$ iff S is either a scroll over a curve of genus $q = \pi$, or $\mathbb{P}^2$ linearly embedded, or the Veronese surface.
(II) If $\mid H+K \mid \neq \emptyset$, then $\mid H+K \mid$ is base point free. In this case
a) $(H+K)^2 = 0$ iff S is a Del Pezzo surface or a conic bundle,
b) If $(H+K)^2 > 0$, then the map $\varphi_{H+K} : S \longrightarrow S' \subset \mathbb{P}^N$ is birational onto a smooth surface S' and is the blowing down of the (-1) lines on S unless
1) $S = \mathbb{P}^2(p_1,\ldots,p_7)$ and $H \equiv 6l - \sum_{i=1}^{7} 2E_i$ (the Geiser involution)
2) $S = \mathbb{P}^2(p_1,\ldots,p_8)$ and $H \equiv 6l - \sum_{i=1}^{7} 2E_i - E_8$
3) $S = \mathbb{P}^2(p_1,\ldots,p_8)$ and $H \equiv 9l - \sum_{i=1}^{8} 3E_i$ (the Bertini involution)
4) $S = \mathbf{P}(E)$, where E is a rank 2 indecomposable bundle over an elliptic curve and $H \equiv 3B$ where B is a section with $B^2 = 1$ on S.

Related to the above theorem, the following two results impose restrictions on the invariants of a smooth surface in $\mathbb{P}^4$.

Theorem 0.14. [Au1][La]. *The only smooth, non-degenerate scrolls in $\mathbb{P}^4$ are the rational cubic and the elliptic quintic scrolls.*

Theorem 0.15. [ES]. *The only smooth conic bundles in $\mathbb{P}^4$ are the rational Castelnuovo surfaces of degree 5 and the Del Pezzo surfaces of degree 4.*

We derive some easy consequences which will be freely used in the text:

Lemma 0.16. *Let S be a smooth surface with $\kappa(S) \geq 0$ and let H be a very ample divisor. Then $(H+K)K \geq 0$.*

Proof. Clear since K is pseudo-effective.□

Lemma 0.17. *Let S be a smooth surface, let H be a very ample divisor on it and assume that $\pi(S) \neq q(S)$. Then $\mid K - H \mid = \emptyset$, when $K^2 < H^2$.*

Proof. In the above hypothesis $\mid K + H \mid$ is nef.□

Lemma 0.18. *Let S be a smooth surface and let H be a very ample divisor on it. Then $(HK)^2 < H^2$ implies that $\kappa(S) \leq 1$.*

Proof. We observe first that the above numerical condition implies, via the Hodge index theorem, that $K^2 \leq 0$. Assume now that S would be a surface of general type and consider the adjunction morphism

$$\varphi_{H+K} : S \longrightarrow S_1 \overset{H_1}{\hookrightarrow} \mathbb{P}^N,$$

which blows down the (-1) lines on S. By using deg $S_1 = H_1^2 = (H+K)^2$ and $H_1K_1 = (H+K)K$ we obtain

$$H_1^2 = (H^2 + 2HK + K^2) \;>\; (HK)^2 + 2(HK)K^2 + (K^2)^2 = (HK + K^2)^2 = (H_1K_1)^2$$

since $2HK + K^2 = (2H+K)K \geq 0$, $K^2 < 1$, and by assumption $(HK)^2 < H^2$. This means that the numerical hypothesis is preserved through the adjunction process, hence it will hold also on the minimal model $S_{\min}$ of S, where the adjunction stops. Therefore we would obtain $K^2_{\min} \leq 0$ which is impossible for a minimal surface of general type. □

The same arguments prove also the following:

Remark 0.19. *Let S be a smooth surface of general type and let H be a very ample divisor. If $(HK)^2 < pH^2$ for some $p \in \mathbb{N}^*$, then $K^2_{\min} < p$.*□

Lemma 0.20. *Let S be a smooth non-degenerate surface in $\mathbb{P}^4$ with $\kappa(S) = -\infty$. Then either*
a) S is a scroll ($d = 3, 5$), or
b) S is a conic bundle ($d = 4, 5$), or
c) S is a Veronese surface ($d = 4$), or
d) the following inequalities hold

(0.21.) $$5\chi \geq 2\pi - \frac{d^2 - 7d + 8}{2}$$

and

(0.22.) $$7\chi \geq 2\pi - \frac{d^2 - 7d + 4}{2}$$

Moreover, when the last inequality is strict then also

$$26\chi \geq 7\pi - \frac{3d^2+21d-14}{2} \tag{0.23.}$$

Proof. The above inequalities express the fact that the image of S through the adjunction mapping $S_1 = \varphi_{H+K}(S) \subset \mathbb{P}^{\pi+\chi-2}$ is a non-degenerate surface which has sectional genus $\pi_1 \geq q_1$, and that the adjoint linear system on S_1 is globally generated when $\pi_1 \neq q_1$.□

(0.24.) Liaison.(see [PS]) Two surfaces S and S' in $\mathbb{P}^4$ are said to be (geometrically) linked (m,n) if there exist hypersurfaces V and V' of degree n and respectively m such that $V \cap V' = S \cup S'$. There are two standard exact sequences of linkage, namely:

$$0 \longrightarrow \mathcal{O}_S(K) \longrightarrow \mathcal{O}_{S\cup S'}(m+n-5) \longrightarrow \mathcal{O}_{S'}(m+n-5) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_S(K) \longrightarrow \mathcal{O}_S(m+n-5) \longrightarrow \mathcal{O}_{S\cap S'}(m+n-5) \longrightarrow 0.$$

The first sequence yields the relation between the Euler-Poincaré characteristics:

$$\chi(S') = \chi(V \cap V') - \chi(\mathcal{O}_S(m+n-5)). \tag{0.25.}$$

The corresponding sequence for linkage of curves in $\mathbb{P}^3$ yields the following relation between the sectional genera:

$$\pi(S) - \pi(S') = \frac{1}{2}(m+n-4)(d(S)-d(S')). \tag{0.26.}$$

In terms of syzygies if

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{I}_S \longrightarrow 0$$

is a resolution with locally free sheaves of the ideal sheaf $\mathcal{I}_S$ of S, then a mapping cone produces a locally free resolution

$$0 \longrightarrow \mathcal{E}_0^\vee(-m-n) \longrightarrow \mathcal{E}_1^\vee(-m-n) \oplus \mathcal{O}(-m) \oplus \mathcal{O}(-n) \longrightarrow \mathcal{I}_{S'} \longrightarrow 0$$

for the ideal sheaf of the linked surface. In particular we obtain the following isomorphisms between the Hartshorne-Rao modules

$$M^{3-i}(S) \cong M^i(S')^*(5-m-n) \qquad i=1,2 \tag{0.27.}$$

We recall also the relation between the homogeneous ideals of the linked surfaces

$$I_{S'} = (I_{V\cap V'} : I_S) \tag{0.28.}$$

For a proof of existence via linkage, the following propositions will be used:

Proposition 0.29. *If S and T are linked, then S is locally Cohen-Macaulay if and only if T is locally Cohen-Macaulay.*

See [PS, Proposition 1.3] for a proof.

Proposition 0.30. *If Z is a local complete intersection surface in $\mathbb{P}^4$, scheme theoretically cut out by hypersurfaces of degree d, then Z can be linked to a smooth surface S in the complete intersection of two hypersurfaces of degree d.*

For a proof see [PS, Proposition 4.1].

Remark 0.31. *(*Peskine, cf. [Ra1]*) The above proposition remains true if Z is not a local complete intersection in at most a finite set of points but it is locally Cohen-Macaulay and the tangent cone at each of these points is linked to a plane in a complete intersection.*

In order to determine the minimal elements of an even liaison class we will need the following version of a lemma from [LR]:

Lemma 0.32. *Let Z be a codimension two, locally Cohen-Macaulay subscheme of $\mathbb{P}^n$ and define the speciality of Z as $e(Z) := \max\{\, t \mid h^{n-2}(\mathcal{O}_Z(t)) \neq 0 \,\}$.*
a) If $h^0(\mathcal{I}_Z(e(Z)+n)) = 0$, then Z is a minimal element in its even liaison class.
b) If moreover $h^0(\mathcal{I}_Z(e(Z)+n+1)) = 0$, then Z is the unique minimal element in its even liaison class.

The proof in [LR] can be easily adapted to the general case.

(0.33.) Reducible curves. One method to work out the classification of the various studied surfaces will be to find special reducible hyperplane sections on them. For the discussion of the components with an arithmetic genus too high in comparison with their degree we use the following

Lemma 0.34. *Let C be a curve of degree d and arithmetic genus p on a smooth surface S in $\mathbb{P}^4$. One has the following possibilities:*
a) If $d \leq 3$, then $p \leq 1$ with equality iff C is a plane cubic curve
b) If $d = 4$, then $p \leq 1$ or $p = 3$ and C is a plane quartic curve
c) If $d = 5$, then $p \leq 3$ or $p = 6$ and C is a plane quintic curve
d) If $d = 6$, then $p \leq 6$ or $p = 10$ and C is a plane sextic curve. Moreover, if $p = 5$ (resp. $p = 6$) then C decomposes into a plane quintic curve and a line which don't meet (resp. meet in a point)
e) If $d = 7$, then $p \leq 6$ unless C is a plane septic curve, or C decomposes into a plane sextic and a line which meet in a point ($p = 10$), or which don't meet ($p = 9$), or C decomposes into a plane quintic curve and a plane conic which meet along a subscheme of length two ($p = 7$)
f) If $d = 8$, then $p \leq 9$ unless C is a plane curve, or C decomposes into a plane septic and a line which meet in a point ($p = 15$), or which don't meet ($p = 14$), or C decomposes into a plane sextic curve and a plane conic which meet along a subscheme of length two

$(p = 11)$, or in a point $(p = 10)$, or C decomposes into a plane sextic curve and two skew lines meeting the sextic in points $(p = 10)$.

Proof. See [Ra1]. It is straightforward using Castelnuovo's bound for the genus of an irreducible curve and formula (0.2). For the last case one can use also the results of [GLP] relating the regularity of the ideal of an irreducible curve with the existence of higher order secant lines.□

Lemma 0.35. *Let $S \subset \mathbb{P}^4$ be a smooth non-degenerate surface and let π be a plane in $\mathbb{P}^4$ cutting S along a curve C. Then C has no multiple components.*

Proof. Assume that C would have a multiple component D, i.e. $C = 2D + E$ with E effective and let $d = \deg D$. Then adjunction gives

$$D^2 + DK = (d-1)(d-2) - 2 = d^2 - 3d$$

and

$$2D(2D + K) = (2d-1)(2d-2) - 2.$$

It follows that $D^2 = d$ which contradicts Hodge index

$$D^2 \leq DH/\deg S < d. \qquad \square$$

Lemma 0.36.[PR]. *Assume now that $S \subset \mathbb{P}^4$ contains a plane curve of degree d_p. Then*
a) $\mathrm{h}^1(S, \mathcal{O}_S(1)) \geq \frac{1}{2}(d_p - 2)(d_p - 3)$ if $p_g = 0$, while
b) $\mathrm{h}^1(S, \mathcal{O}_S(1)) \geq \frac{1}{2}(d_p - 2)(d_p - 3) + 1 - p_g$ if $p_g \geq 1$.

Proof. Let C be the plane curve on S, let $D = H - C$ and consider the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(H) \longrightarrow \mathcal{O}_C(H) \longrightarrow 0$$

Then $\mathrm{h}^2(S, \mathcal{O}_S(D)) = 0$ if $p_g = 0$, hence $\mathrm{h}^1(S, \mathcal{O}_S(H)) \geq \mathrm{h}^1(C, \mathcal{O}_C(H))$. In general $\mathrm{h}^2(S, \mathcal{O}_S(D)) \leq p_g - 1$ when $p_g \geq 1$, so $\mathrm{h}^1(S, \mathcal{O}_S(H)) \geq \mathrm{h}^1(C, \mathcal{O}_C(H)) - p_g + 1$.□

The previous bounds will also be used in connection with the following:

Lemma 0.37.[Bo]. *Let C be an effective Cartier divisor on a smooth surface S and let L be a line bundle on C with $\mathrm{h}^0(C, L) \neq 0$. Then there exists a decomposition $C = C_1 + C_2$, $C_1 \geq 0$, $C_2 > 0$ such that*

$$C_1 C_2 \leq \deg_{C_2}(L_{|C_2}).$$

The following lemmas will be used in the text to produce reducible curves on the studied surfaces.

Lemma 0.38.[Ra2]. *Let S be a smooth, (proper) elliptic surface in $\mathbb{P}^4$ and let S_{min} be its minimal model. We denote also by $m = \min\{-K^2, p_g - 1\}$.*

a) If $p_g = 1$, then $HK + K^2 \geq 3$.
b) If $p_g \geq 2$, then

$$\frac{HK + K^2 - m}{p_g - 1} \geq 3.$$

Proof. See [Ra2]. One uses Kodaira's formula [BPV] for the canonical class of an elliptic fibration and the fact that an elliptic curve has degree at least 3. □

Analogously, for general-type surfaces one has

Lemma 0.39.[Ra2]. *Let S be a smooth general type surface in $\mathbb{P}^4$ with minimal model S_{min}. We denote by K_{min} the total transform on S of the canonical class on S_{min} and by $m = \min\{K^2_{min} - K^2, p_g - 1\}$. Then there exists a curve D on S of degree*

$$\deg D \leq HK + K^2 - K^2_{min} - m$$

and arithmetic genus

$$p_a(D) = K^2_{min} + 1.$$

(0.40.) Postulation of points. When identifying 6-secant lines to surfaces in $\mathbb{P}^4$ we'll look to their plane sections through such lines and we'll make use of the following result from [Ra1].

Lemma 0.41. *Let $\pi : S \to \mathbb{P}^2$ be the morphism obtained by blowing up t points (some possibly infinitely close) in $\mathbb{P}^2$, where $9 \leq t \leq 12$. Denote the exceptional divisors by $E_1, ..., E_t$ and consider the linear system*

$$|C| = |4\pi^* l - \sum_{i=1}^{t} E_i|$$

on S.

If $\dim|C| \geq 15 - t$ and $|C|$ has a fixed curve, then there is a curve $\Gamma \equiv \pi^* l - \sum_{k=1}^{6} E_{i_k}$ or $\Gamma \equiv 2\pi^* l - \sum_{k=1}^{10} E_{i_k}$ on S, which is part of the fixed curve of $|C|$.
If $\dim|C| \geq 15 - t$ and $|C|$ has no fixed curve, then $t = 12$, $\dim|C| = 3$ and $|C|$ has no base points. Furthermore there is a curve $\Gamma \equiv 3\pi^* l - \sum_{i=1}^{12} E_i$ on S.
If $t = 12$, $\dim|C| = 2$ and $|C|$ has a fixed curve, then there is a curve $\Gamma \equiv \pi^* l - \sum_{k=1}^{5} E_{i_k}$ or $\Gamma \equiv 2\pi^* l - \sum_{k=1}^{9} E_{i_k}$ or $\Gamma \equiv 3\pi^* l - \sum_{i=1}^{12} E_i$ on S, which is part of the fixed curve of $|C|$.
If $t = 12$, $\dim|C| = 2$ and $|C|$ has no fixed curve, then $|C|$ has at the most one basepoint.

For a proof see [Ra1], ($t = 12$), and [PR]. The above lemma is, in a certain sense, a special case of the general result in [EP].

1. The Eagon-Northcott complex method

We recall in this chapter a powerful method of construction for surfaces in $\mathbb{P}^4$ which was first introduced in [DES]. The rough idea is to realize the surface one wishes to construct as the determinantal locus of a morphism between two vector bundles on $\mathbb{P}^4$. Namely, let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $\mathbb{P}^4$ of ranks $\mathrm{rk}\mathcal{E} = r$, $\mathrm{rk}\mathcal{F} = r+1$ and, for $\varphi \in \mathrm{Hom}(\mathcal{E}, \mathcal{F})$ an injective morphism, consider the degeneracy locus $\mathrm{D}(\varphi)$ which is the subscheme given locally by the r-minors of the matrices associated to φ. As a set

$$\mathrm{D}(\varphi) = \{\, x \in \mathbb{P}^4 \;\mid\; \mathrm{rk}\,\varphi(x) \leq r-1 \,\}.$$

General facts, see [ACGH], implies that $\mathrm{D}(\varphi)$, when non-empty, has codimension ≥ 2. Moreover, when it has the expected codimension, i.e. equality holds, $\mathrm{D}(\varphi)$ is a locally Cohen-Macaulay surface. Given a morphism φ, one constructs the so called Eagon-Northcott complex:

$$\begin{tikzcd}
\wedge^{r+1}\mathcal{F}^* \otimes \wedge^r\mathcal{E} \arrow[r, "\wedge^r\varphi^*(c_1(\mathcal{E})-c_1(\mathcal{F}))"] \arrow[d, "\mathrm{id}\otimes\wedge^r\varphi"] & \mathcal{F}^* \arrow[r, "\varphi^*"] & \mathcal{E}^* \\
\wedge^{r+1}\mathcal{F}^* \otimes \wedge^r\mathcal{F} \arrow[r, "\text{contraction}"] & \mathcal{F}^* \arrow[u, "\mathrm{id}"] &
\end{tikzcd}.$$

Proposition 1.1. *In the above notations, the following statements are equivalent:*
a) coker φ is an ideal sheaf.
b) coker φ is either the ideal sheaf of a pure codimension 2 subscheme or $\mathcal{O}_{\mathbb{P}^4}$.
c) The Eagon-Northcott complex

$$0 \longrightarrow \mathcal{E} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\wedge^r\varphi(c_1(\mathcal{F})-c_1(\mathcal{E}))} \mathcal{O}_{\mathbb{P}^4}(c_1(\mathcal{F}) - c_1(\mathcal{E}))$$

is exact.

Proof. See [ACGH].□

For later reference, we state here two well known corollaries of the above proposition:

Corollary 1.2. *Let S be a reduced scheme, let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $\mathbb{P}^4 \times S$ of ranks r and respectively $r+1$ and let $\varphi : \mathcal{E} \longrightarrow \mathcal{F}$ be an injective morphism. Let also $Z = \mathrm{D}(\varphi)$ and suppose that Z dominates S. Then the set of points where $Z_s \subset \mathbb{P}^4 \times \{s\}$ has pure codimension two is open and Z is flat over S above it.*

Corollary 1.3. *Let $\mathcal{E}$ and $\mathcal{F}$ be two vector bundles on $\mathbb{P}^4$ of ranks r and respectively $r+1$ and let φ_1 and φ_2 be two morphisms between $\mathcal{E}$ and $\mathcal{F}$. Assume also that both determinantal loci $S_i = \mathrm{D}(\varphi_i)$, $i = \overline{1,2}$ have the expected codimension two. Then S_1 and S_2 lie in the same irreducible component of the Hilbert scheme.*

For a proof of both corollaries see for example [BB] or [MDP].

The aim of the method is to realize the desired surface S as a, hopefully smooth, degeneration locus of a morphism φ between suitable chosen vector bundles $\mathcal{E}$ and $\mathcal{F}$. In this case, the Eagon-Northcott complex provides a vector bundle resolution for the ideal sheaf of the surface and we can relate the invariants of the surface S with those of $\mathcal{E}$ and $\mathcal{F}$. Normalizing, it is is enough to deal with the case of an exact sequence of type:

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_S \longrightarrow 0$$

where $\mathcal{E}$, $\mathcal{F}$ are as above and S is a surface in $\mathbb{P}^4$.
Riemann-Roch without denominators [Fu], gives

$$1 - i_*(c(N^*_{S|\mathbb{P}^4})^{-1}) = c(\mathcal{E} - \mathcal{F})$$

where $i : S \hookrightarrow \mathbb{P}^4$ is the inclusion and $\mathcal{E} - \mathcal{F}$ means difference in the Grothendieck group.
The homogeneous parts yields the following relations:
in degree 2, Porteus´ formula

$$\deg S = d = c_2(\mathcal{F}) - c_2(\mathcal{E})$$

in degree 3

$$i_*(c_1(N_{S|\mathbb{P}^4})) = c_3(\mathcal{F}) - c_3(\mathcal{E}) - c_1(c_2(\mathcal{F}) - c_2(\mathcal{E})) \qquad (*)$$

in degree 4

$$i_*(c_1(N_{S|\mathbb{P}^4})^2) = c_4(\mathcal{F}) - c_4(\mathcal{E}) - c_1(c_3(\mathcal{F}) - c_3(\mathcal{E})) + (c_1^2 - c_2(\mathcal{E}))(c_2(\mathcal{F}) - c_2(\mathcal{E})) \quad (**)$$

where $c_1 = c_1(\mathcal{E}) = c_1(\mathcal{F})$.
Assume now that S is smooth. Then adjunction and the self-intersection formula [Ha] give $c_1(N_{S|\mathbb{P}^4}) = 5H + K$ and $c_2(N_{S|\mathbb{P}^4}) = d^2$. Substituting in $(*)$ and $(**)$ one obtains for the sectional genus

$$2\pi - 2 = c_3(\mathcal{F}) - c_3(\mathcal{E}) - d(4 + c_1)$$

and for the self-intersection of the canonical class

$$K^2 = c_4(\mathcal{F}) - c_4(\mathcal{E}) - (10 + c_1)(c_3(\mathcal{F}) - c_3(\mathcal{E})) + (c_1^2 - c_2(\mathcal{E}))(c_2(\mathcal{F}) - c_2(\mathcal{E})) + (25 + 10c_1)d.$$

Finally the double point formula, for instance, allows to determine the Euler characteristic $\chi(\mathcal{O}_S)$.

To construct a surface S with the desired invariants we'll have to find appropriate vector bundles $\mathcal{E}$ and $\mathcal{F}$. A general way to construct vector bundles (or more generally coherent sheaves) is to determine first the differentials of the Beilinson's spectral sequence:

Theorem 1.4.[Bei]. *Let $\mathcal{G}$ be a coherent sheaf on $\mathbb{P}^n = \mathbf{P}(V)$. There exists a spectral sequence with E_1 terms*

$$E_1^{pq} = \mathrm{H}^q(\mathbb{P}^n, \mathcal{G}(p)) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)$$

converging to $\mathcal{G}$; i.e. $E_\infty^{pq} = 0$ for $p+q \neq 0$ and $\oplus E_\infty^{p,-p}$ is the associated graded sheaf of a suitable filtration of $\mathcal{G}$.

All the E_1-terms are in the 2^{nd} quadrant and only finitely many of them are non-zero. Via the canonical isomorphisms induced by contraction, $\mathrm{Hom}(\Omega^i_{\mathbb{P}^n}(i), \Omega^j_{\mathbb{P}^n}(j)) \cong \Lambda^{i-j}V$, for $i \geq j$, cf. [Bei], the d_1-differentials

$$d_1^{pq} \in \mathrm{Hom}(\mathrm{H}^q(\mathbb{P}^n, \mathcal{G}(p)) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p), \mathrm{H}^q(\mathbb{P}^n, \mathcal{G}(p+1)) \otimes \Omega_{\mathbb{P}^n}^{-p-1}(-p-1))$$

can be identified with the natural multiplication maps in

$$\mathrm{Hom}(\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathrm{H}^q(\mathbb{P}^n, \mathcal{G}(p)), \mathrm{H}^q(\mathbb{P}^n, \mathcal{G}(p+1)).$$

This also means that to determine the d_1-differentials is equivalent to fixing the module structure of $\oplus_p \mathrm{H}^q(\mathbb{P}^n, \mathcal{G}(p))$. The higher order differentials d_r are induced by maps $E_1^{pq} \longrightarrow E_1^{p+r,q-r+1}$. These maps are not canonically given.

The method is to apply Beilinson's spectral sequence to the twisted ideal sheaf $\mathcal{I}_S(m)$ for $m = 3$ or 4 and to interpret part of the spectral sequence as the spectral sequence for a vector bundle $\mathcal{E}$ and the other part as that for a vector bundle $\mathcal{F}$. The differentials will induce a morphism

$$\varphi : \mathcal{E} \longrightarrow \mathcal{F}$$

whose degeneracy locus $\mathrm{D}(\varphi)$ will be the desired surface.
The E_1-terms of the spectral sequence are determined by the dimensions $\mathrm{h}^i(\mathcal{I}_S(p))$ for $i = \overline{0,4}$, $p = \overline{m-4, m}$. Riemann-Roch gives

Lemma 1.5. *Let S be a smooth surface in $\mathbb{P}^4$. Then*

$$\chi(\mathcal{I}_S(p)) = \binom{p+4}{4} - \binom{p+1}{2} d + p(\pi - 1) - \chi.$$

We observe also that some of the cohomology groups vanish :

- $\mathrm{h}^4(\mathcal{I}_S(p)) = \mathrm{h}^4(\mathcal{O}_{\mathbb{P}^4}(p)) = 0$ for all $p \geq -4$;
- $\mathrm{h}^3(\mathcal{I}_S(p)) = \mathrm{h}^2(\mathcal{O}_S(p)) = \mathrm{h}^0(\mathcal{O}_S(K - pH)) = 0$ for all $p \geq 1$,
  when, for example, either S is not of general type or S is a general type surface with $p_g \leq 4$, or S is a general type surface with $K^2 < HK$;
- $\mathrm{h}^2(\mathcal{I}_S(p)) = \mathrm{h}^1(\mathcal{O}_S(p)) = \mathrm{h}^1(\mathcal{O}_S(K - pH)) = 0$ for all $p \leq -1$
  by Kodaira's vanishing theorem;
- $\mathrm{h}^1(\mathcal{I}_S(p)) = 0$ for all $p \leq 0$;
- $\mathrm{h}^1(\mathcal{I}_S(1)) = 0$,
  by (0.7), unless if S is the Veronese surface;
- $\mathrm{h}^0(\mathcal{I}_S(1)) = 0$,
  unless if S is degenerate, i.e. lies in a hyperplane.

In particular the speciality, cf. [Al1], of a smooth non-degenerate, non-general type surface different from the Veronese surface is given by Riemann-Roch

$$s = \mathrm{h}^1(\mathcal{O}_S(1)) = \chi(\mathcal{I}_S(1)) = \pi - d + 4 - \chi$$

Related to the postulation of a surface in $\mathbb{P}^4$ we mention the following results:

Theorem 1.6.[Ro][Au1]. *Let S be a smooth non-degenerate surface in $\mathbb{P}^4$ which is contained in a hyperquadric. Then*

$$\pi = 1 + \left[\frac{d(d-4)}{4}\right]$$

and S is either the complete intersection of the hyperquadric with another hypersurface, or S is linked to a plane in the complete intersection of the hyperquadric with another hypersurface.

Theorem 1.7.[Au1][Ko]. *Let S be smooth surface in $\mathbb{P}^4$ which is contained in an irreducible cubic hypersurface V. Then either S is projectively Cohen-Macaulay and linked on V to an irreducible scheme of degree ≤ 3, or S is linked on V to a Veronese surface, or to an elliptic quintic scroll.*

Corollary 1.8. *A smooth surface S in $\mathbb{P}^4$ of degree ≥ 9 which is contained in a cubic hypersurface is necessarily of general type.*

Since surfaces on quadric and cubic hypersurfaces are completely classified, for construction purposes we will mainly assume that

$$\mathrm{h}^0(\mathcal{I}_S(2)) = \mathrm{h}^0(\mathcal{I}_S(3)) = 0.$$

To conclude, for a smooth non-degenerate surface S in $\mathbb{P}^4$, not of general type, different from the Veronese surface and not lying on a cubic hypersurface, we can draw the following Beilinson cohomology diagram:

$$\begin{array}{c|c|c|c|c|c|}
 & \uparrow i & & & & \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
\hline
N+1 & p_g & 0 & 0 & 0 & 0 \\
\hline
0 & q & s & \mathrm{h}^1(\mathcal{O}_S(2)) & & \\
\hline
0 & 0 & 0 & \mathrm{h}^1(\mathcal{I}_S(2)) & \mathrm{h}^1(\mathcal{I}_S(3)) & \mathrm{h}^1(\mathcal{I}_S(4)) \\
\hline
0 & 0 & 0 & 0 & 0 & \mathrm{h}^0(\mathcal{I}_S(4)) \\
\hline
\end{array} \xrightarrow{\quad} p \quad ,$$

where

$$N = \pi + \chi - 2.$$

As previously mentioned, to construct the surface S we will first try to identify the Hartshorne-Rao modules $\mathrm{H}^i(\mathbb{P}^4, \mathcal{I}_S(*))$, $i = 1, 2$. Sheafification of suitable syzygy modules will provide afterwards the needed vector bundles $\mathcal{E}$ and $\mathcal{F}$.

(1.9.) Graded artinian modules. We will briefly recall here some basic facts about graded, finite length modules. Let $V = \mathrm{span}_k < e_0, e_1, e_2, e_3, e_4 >$. Let also $R = k[x_0, x_1, x_2, x_3, x_4]$ be the homogeneous coordinate ring of $\mathbb{P}^4 = \mathbb{P}(V)$ and let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a graded k-vector space of finite length. A structure of a graded R-module on M is determined by a degree 0, graded morphism of k-algebras $f : R \to \mathrm{Endgr}(M)$, i.e., by a set of k-linear morphisms

$$f_d : R_d \to \mathrm{End}^d(M) = \{ \ u \in \mathrm{End}(M) \ \mid \ u(M_i) \subset M_{i+d} \ \forall \ i \ \},$$

for $d \in \mathbb{N}$. More precisely, since $R = S^*(V^*)$, this is equivalent to fix a degree 0, graded k-morphism

$$u : R_1 \otimes_k M(-1) \to M \tag{$*$}$$

satisfying the following commutation relations:

$$u_{i+1}(h \otimes u_i(g \otimes \alpha)) = u_{i+1}(g \otimes u_i(h \otimes \alpha))$$

for all i, $g, h \in R_1 = V^*$, $\alpha \in M_i$, where we have denoted by $u_i : R_1 \otimes_k M_i \to M_{i+1}$ the degree i component of u.

We fix for the sequel on $M = \oplus_{n\in\mathbb{Z}} M_n$ a finite length, graded R-module structure. We'll denote by

$$M^\vee = \mathrm{Hom}_R(M, R)$$

the dual module and by

$$M^* = \mathrm{Hom}_k(M, k) = \underset{n\in\mathbb{Z}}{\oplus} \mathrm{Hom}_k(M_{-n}, k)$$

the k-dual module. We recall also that

$$M^* \cong \mathrm{Hom}_R(M, R^*) \tag{$*$}$$

and that local duality gives

$$\mathrm{Ext}^i_R(M, R) = 0, \quad \text{for } i \leq 4 \quad \text{and} \quad \mathrm{Ext}^5_R(M, R) \cong M^*(5)$$

the isomorphisms being homogeneous of degree 0. Let

$$0 \longleftarrow M \xleftarrow{\sigma_0} L_0 \xleftarrow{\sigma_1} L_1 \xleftarrow{\sigma_2} L_2 \longleftarrow \cdots \longleftarrow L_4 \xleftarrow{\sigma_5} L_5 \longleftarrow 0 \tag{$*$}$$

be a minimal free resolution of M. We distinguish, for later reference, the following data:

- the support of M

supp $M = \{ \ n \ \mid \ M_n \neq 0 \ \}$

- the Hilbert function

$\chi_{_M}(n) = \dim_k M_n = \sum_{i=0}^{5} (-1)^i \mathrm{h}^0(\mathbb{P}^4, \widetilde{L_i}(n))$

- the least degree of a generator of M

$l_{_M} = \min\{ \ n \in \mathbb{Z} \ \mid \ M_n \neq 0 \ \} = \min\{ \ n \in \mathbb{Z} \ \mid \ \mathrm{h}^0(\mathbb{P}^4, \widetilde{L_0}(n)) \neq 0 \ \}$

- the highest degree of a generator of $L_5 \ -5$

$s_{_M} = \max\{ \ n \in \mathbb{Z} \ \mid \ M_n \neq 0 \ \} = -5 \ + \ \max\{ \ n \in \mathbb{Z} \ \mid \ \mathrm{h}^0(\mathbb{P}^4, \widetilde{L_5}^*(-n)) \neq 0 \ \}$

- the Mumford-Castelnuovo regularity of the sheafified syzygies

$e(Syz_k(M)) = \max\limits_{\substack{i,j \\ j \geq k+1}} \ (e_{ij} - j) \ \text{ if } \ L_i = \bigoplus\limits_{j=1}^{\mathrm{rk}\, L_i} R(-e_{ij}) \ \ , \ i = \overline{0,5},$

where we have denoted by

$$Syz_k(M) := (\ker \ \sigma_k)^{\sim} = (\mathrm{im} \ \sigma_{k+1})^{\sim} \ , \qquad i = \overline{1,4}$$

the k-th sheafified syzygy module.
A minimal free resolution of M^* is obtained by taking the R-dual of $(*)$:

$$0 \longleftarrow M^*(5) \xleftarrow{\theta} L_5^{\vee} \xleftarrow{\sigma_5^{\vee}} L_4^{\vee} \xleftarrow{\sigma_4^{\vee}} L_3^{\vee} \longleftarrow \ \cdots \ \longleftarrow L_1^{\vee} \xleftarrow{\sigma_1^{\vee}} L_0^{\vee} \longleftarrow 0$$

We recall now how to recover a minimal free resolution of M from the knowledge of the multiplication map $u : R_1 \otimes_k M(-1) \to M$. In Green's terms [Gr], this is to compute the Koszul homology groups of M. Tensoring the resolution $(*)$ by $k = R/(x_0, x_1, x_2, x_3, x_4)$ over R we obtain a complex of k-vector spaces

$$0 \longleftarrow \overline{M} \xleftarrow{\overline{\sigma}_0} \overline{L}_0 \xleftarrow{\overline{\sigma}_1} \overline{L}_1 \xleftarrow{\overline{\sigma}_2} \overline{L}_2 \longleftarrow \ \cdots \ \longleftarrow \overline{L}_4 \xleftarrow{\overline{\sigma}_5} \overline{L}_5 \longleftarrow 0$$

whose homology yields the $\mathrm{Tor}^i_R(M,k)$'s. Moreover, the resolution $(*)$ being minimal, we obtain that in fact $\overline{L}_i \cong \mathrm{Tor}^i_R(M,k)$ as graded k-vector spaces, whence

$$L_i = \bigoplus_{j \in \mathbb{Z}} \left(\mathrm{Tor}^i_R(M,k)\right)_j \otimes_k R(-j-i) \qquad \text{for all } i = \overline{0,5}.$$

To compute the Tor-modules one may use also the resolution of k given by the Koszul complex $K_{\bullet}$

$$0 \longleftarrow k \xleftarrow{d_0} R \xleftarrow{d_1} \Lambda^1 V^* \otimes_k R(-1) \xleftarrow{d_2} \Lambda^2 V^* \otimes_k R(-2) \longleftarrow \ \cdots \ \xleftarrow{d_5} \Lambda^5 V^* \otimes_k R(-5) \longleftarrow 0.$$

Tensoring $K_\bullet$ over R with M yields the complex $K_\bullet \otimes_k M$

$$0 \longleftarrow M \xleftarrow{\delta_1} \Lambda^1 V^* \otimes_k M(-1) \xleftarrow{\delta_2} \Lambda^2 V^* \otimes_k M(-2) \longleftarrow \cdots \xleftarrow{\delta_5} \Lambda^5 V^* \otimes_k M(-5) \longleftarrow 0$$

which computes the $\mathrm{Tor}^i_R(M,k)$'s. We observe also that the differential δ_1 is nothing else than the multiplicative structure map $u : R_1 \otimes_k M(-1) \to M$. To bring everything together and to compute also the morphisms appearing in the minimal resolution of M we use a double complex of graded R-modules, whose terms are

$$C_{i,j} = M_{t-j} \otimes_k \overset{i+j-t}{\Lambda} V^* \otimes_k R(-i), \qquad i, j \geq 0,$$

where, for simplicity, we assumed that M is normalized such as $M = \oplus_{i=0}^t M_i$ and that, by convention, all exterior powers outside the range $0, \dots, 5$ vanish. As for the differentials the vertical ones are given by $\delta = (\delta_{i+j-t})_i \otimes_k 1\!\!\mathrm{I}_{\mathrm{R}}(-\mathrm{i}) : \mathrm{C_{i,j}} \to \mathrm{C_{i,j-1}}$ where, as above, δ_p are the differentials of $K_\bullet \otimes_k M$, and the horizontal ones by $d = 1\!\!\mathrm{I}_{\mathrm{M_{t-j}}} \otimes_{\mathrm{k}} (\mathrm{d_{i+j-t}}(\mathrm{j}-\mathrm{t})) : \mathrm{C_{i,j}} \to \mathrm{C_{i-1,j}}$. We set as usual $D = d + \delta$.

Lemma 1.10. $(d+\delta)^2 = 0$.

Proof. Clear.□

To summarize, we have the following display of the double complex $C_{\bullet\bullet}$:

$$\begin{array}{ccccc}
\downarrow & & \downarrow & & \downarrow \\
M_0 \otimes_k R & \xleftarrow{1\!\!\mathrm{I}_{M_0} \otimes \mathrm{d}_1} & M_0 \otimes_k \Lambda^1 V^* \otimes_k R(-1) & \xleftarrow{1\!\!\mathrm{I}_{M_0} \otimes \mathrm{d}_2} & M_0 \otimes_k \Lambda^2 V^* \otimes_k R(-2) \longleftarrow \cdots \\
\downarrow & & \downarrow {\scriptstyle (\delta_1)_0 \otimes 1\!\!\mathrm{I}_{\mathrm{R}(-1)}} & & \downarrow {\scriptstyle (\delta_2)_0 \otimes 1\!\!\mathrm{I}_{\mathrm{R}(-2)}} \\
0 & \longleftarrow & M_1 \otimes_k R(-1) & \xleftarrow{1\!\!\mathrm{I}_{M_1} \otimes \mathrm{d}_1} & M_1 \otimes_k \Lambda^1 V^* \otimes_k R(-2) \longleftarrow \cdots \\
\downarrow & & \downarrow & & \downarrow {\scriptstyle (\delta_1)_1 \otimes 1\!\!\mathrm{I}_{\mathrm{R}(-2)}} \\
0 & \longleftarrow & 0 & \longleftarrow & M_2 \otimes_k R(-2) \qquad \longleftarrow \cdots \\
\downarrow & & \downarrow & & \downarrow
\end{array}$$

We remark that, since $\bigoplus_i C_{i,j} = M_{t-j} \otimes_k K^\bullet(j-t)$, the horizontal lines are exact except for the most left, non-zero, positions where the cokernels of the differentials are, as graded R-modules, $M_{t-j} \otimes_k k(j-t)$. This means that the homology spectral sequence associated to the second filtration of the above complex is degenerated at ${}''\mathrm{E}^1$ and therefore we get the following

Theorem 1.11. *The following assertions hold:*
a) The total homology of the double complex $C_{\bullet\bullet}$ is

$$\mathrm{H}^D_n(C_{\bullet\bullet}) = \begin{cases} M & \text{when } n = t \qquad \text{(as } R\text{-modules)} \\ 0 & \text{otherwise} \end{cases}$$

and $\overset{t}{\underset{k=0}{\oplus}}\,{}''\mathrm{E}^{\infty}_{k,t-k} = \overset{t}{\underset{k=0}{\oplus}}\,{}''\mathrm{E}^{1}_{k,t-k}$ *is the associated graded module to the descending filtration by degrees of* M *(i.e. by* $\overset{t}{\underset{n\geq k}{\oplus}} M_n, \quad k \geq 0$*).*

b) The homology spectral sequence associated to the first filtration computes a minimal free resolution of M*, namely*

$$0 \longleftarrow M = \operatorname{coker}\sigma_1 \longleftarrow \overset{t}{\underset{k=0}{\oplus}}\,{}'\mathrm{E}^1_{k,t-k} \xleftarrow{\sigma_1} \overset{t}{\underset{k=0}{\oplus}}\,{}'\mathrm{E}^1_{k,(t+1)-k} \xleftarrow{\sigma_2} \cdots \longleftarrow \overset{t}{\underset{k=0}{\oplus}}\,{}'\mathrm{E}^1_{k,(t+5)-k} \longleftarrow 0$$

where the components of σ_i *are given by the differentials*

$$d^{k-l} : {}'\mathrm{E}^1_{k,(t+i)-k} \longrightarrow {}'\mathrm{E}^1_{l,(t+i-1)-l}, \qquad k, l = \overline{0,t}, \quad k > l$$

with an obvious abuse of notations.

Proof. Clear, except may be for the fact that the R-module structure of $\mathrm{H}^D_t(C_{\bullet\bullet}) = M$ is the one we started with. But this follows easily, since the vertical differentials are induced by the graded pieces of the multiplicative structure morphism $u : R_1 \otimes_k M(-1) \to M$.$\square$

Remark 1.12. *The group* $GL(5,k) = GL(V^*)$ *acts naturally on* $R = S^*(V^*)$ *and on the Koszul complex* $K_\bullet$*, and its action is also compatible with the differentials of the double complex* $C_{\bullet\bullet}$*. In particular, this fact will allow us to identify later the action of subgroups of* $GL(5,k)$ *on the minimal free resolution of a module* M *and to get information on the Betti numbers in case of invariance (e.g., see chapter 7 for an application of this type).*

Remark 1.13. *For practical construction purposes we'll need to determine from the double complex* $C_{\bullet\bullet}$ *only the presentation morphism*

$$M = \operatorname{coker}\sigma_1 \longleftarrow \overset{t}{\underset{k=0}{\oplus}}\,{}'\mathrm{E}^1_{k,t-k} \xleftarrow{\sigma_1} \overset{t}{\underset{k=0}{\oplus}}\,{}'\mathrm{E}^1_{k,(t+1)-k}.$$

Since σ_1 *is a direct sum of differentials* d^n*, this can be done in specific cases by the usual "tic-tac-toe" procedure.*

Remark 1.14. *In the construction of the complex* $C_{\bullet\bullet}$ *one can in fact take for the vertical differentials any lifts of the multiplication maps*

$$(\delta_1)_i : M_i \otimes_k \overset{1}{\Lambda} V^* \otimes_k R(-1) \longrightarrow M_{i+1} \otimes_k R(-1)$$

along the Koszul complexes.

One method we shall use to construct vector bundles with prescribed cohomologies is to sheafify syzygy modules. Namely we use the following

Proposition 1.15. *In the above notation, for all $i = \overline{1,3}$, the sheafified syzygy module $\mathcal{N}_i = Syz_i(M)$ is a vector bundle on $\mathbb{P}^4$ with cohomology*

$$\mathrm{H}^j(\mathbb{P}^4, \mathcal{N}_i(*)) \cong \begin{cases} M & \text{for } j = i, \quad \text{as graded } R\text{-modules} \\ 0 & \text{for all } j \neq i,\ j = \overline{1,3} \end{cases}.$$

Conversely, any vector bundle $\mathcal{F}$ with the above intermediate cohomology is stably equivalent with $\mathcal{N}_i$, namely

$$\mathcal{F} \cong \mathcal{N}_i \oplus \mathcal{L}, \qquad \text{with } \mathcal{L} \text{ a direct sum of line bundles}$$

For a proof, see for instance [DES].

We shall make later implicit use of the following well known result concerning extensions of modules

Proposition 1.16. *Let $\mathcal{E}$ and $\mathcal{F}$ be two coherent sheaves on $\mathbb{P}^4$ and denote by $E = \mathrm{H}^0_*(\mathcal{E})$ and by $F = \mathrm{H}^0_*(\mathcal{F})$. There exists a spectral sequence*

$$\mathrm{E}_2^{pq} = \mathrm{Ext}^p_R(E, \mathrm{H}^q_*(\mathcal{F})) \quad \Rightarrow \quad \mathrm{Ext}^{p+q}_{\mathcal{O}_{\mathbb{P}^4}}(\mathcal{E}, \mathcal{F}).$$

In particular the following comparison sequence is exact

$$0 \longrightarrow \mathrm{Ext}^1_R(E,F)^0 \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^4}}(\mathcal{E},\mathcal{F}) \longrightarrow \mathrm{Hom}_R(E, \mathrm{H}^1_*(\mathcal{F})) \longrightarrow \mathrm{Ext}^2_R(E,F)^0 \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_{\mathbb{P}^4}}(\mathcal{E},\mathcal{F}).$$

Proof. See for instance [MDP]. One writes the spectral sequence of a composition of functors [Gro], and uses the adjunction of the two functors $\widetilde{\ }$ and H^0_*. The comparison sequence is the associated low-degree sequence.□

Remark 1.17.
a) The first map in the comparison sequence is associating to an extension class of an extension of graded R-modules the corresponding extension class in $\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^4}}(\mathcal{E},\mathcal{F})$ of the short exact sequence obtained via sheafification.
b) The second map is associating to the extension class of an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow 0$$

the cobord map $\delta : E \longrightarrow \mathrm{H}^1_*(\mathcal{F})$ in the long exact cohomology sequence.

2. A $K3$ surface of degree 10

We describe in this chapter two constructions of a smooth, non-minimal $K3$ surface $S \subset \mathbb{P}^4$ with $d = 10$, $\pi = 9$. A family of surfaces with these invariants, which all lie on two quartic hypersurfaces, has been constructed by K. Ranestad in [Ra1]. We give an example of a family of surfaces which lie on a single quartic.

In terms of numerical invariants, we recall the following:

Proposition 2.1.[Ra1]. *Let S be a smooth surface of degree 10 in $\mathbb{P}^4$ with $\pi = 9$ and $\chi = 2$. Then either*
a) S is a regular, proper elliptic surface with three (-1) lines, or
b) S is a non-minimal $K3$ surface with three (-1) conics, or
c) S is a non-minimal $K3$ surface with two (-1) lines and one (-1) quartic.

For a proof, see [Ra1] and also [PR] for the non-existence of a smooth $K3$ surface with the above invariants and having one (-1) line, one (-1) conic and one (-1) cubic.

K. Ranestad has constructed in [Ra1] examples of surfaces for the cases a) and b) in the above proposition. We give examples for c). First a lemma.

Lemma 2.2.
a) $\mathrm{h}^1(\mathcal{O}_S(k)) = 0$ *for* $k \geq 2$.
b) $\mathrm{h}^0(\mathcal{I}_S(4)) \geq 1$.

Proof. Severi's theorem gives $\mathrm{h}^1(\mathcal{O}_S(1)) = 1$. Thus if $\mathrm{h}^2(\mathcal{I}_S(2)) > 0$, then $\mathrm{h}^2(\mathcal{I}_H(2)) > 0$ for at least a web of hyperplane sections H. But the general hyperplane section H in the web is smooth and $2\pi - 2 = 16 < 20$, so $\mathcal{O}_H(2H)$ is non-special, i.e., $\mathrm{h}^1(\mathcal{O}_H(2)) = \mathrm{h}^2(\mathcal{I}_H(2)) = 0$, which is a contradiction; hence by induction a). Part a), Riemann-Roch and (1.6), (1.7) give $\mathrm{h}^1(\mathcal{I}_S(2)) = 1$ and $\mathrm{h}^1(\mathcal{I}_S(3)) = 3$, whence $\mathrm{h}^0(\mathcal{I}_H(3)) = 1$ for at least a pencil of hyperplane sections. But if $\mathrm{h}^0(\mathcal{I}_S(4)) = 0$, then $\mathrm{h}^1(\mathcal{I}_S(4)) = 0$ and thus $\mathrm{h}^0(\mathcal{I}_H(4)) = \mathrm{h}^1(\mathcal{I}_S(3)) = 3$ for all H. This is a contradiction since for the hyperplanes in the above pencil $\mathrm{h}^0(\mathcal{I}_H(4)) \geq 4$.□

Therefore the cohomology diagram of the surface S looks like:

i

1				
	1			
		1	3	a
				a

p

$\mathrm{h}^i(\mathcal{I}_S(p))$

where $a \geq 1$. When $a \geq 2$ one shows, see [PR], that in fact $a = 2$ and S is a smooth $K3$ surface with three (-1) conics, as constructed in [Ra1]. A construction of this surface using the Eagon-Northcott complex can be found in [DES].

Therefore we assume in the sequel that $a = 1$ and we look for graded artinian modules $M = \oplus_{n \geq -2} M_n$ with Hilbert function $(1, 3, 1)$, which we'll suppose generated in the first non-zero twist, i.e., monogeneous. Any such module has a minimal free presentation of type

$$0 \leftarrow M \longleftarrow R(2) \xleftarrow{\psi} 2R(1) \oplus 5R$$

where the linear part of ψ is given, without loss of generality, by say x_0 and x_1, and the quadratic by $q_1, q_2, \ldots, q_5 \in k[x_2, x_3, x_4]$, quadrics in three variables without common zeroes. Therefore the choice of ψ is equivalent to that of a hyperplane section of the Veronese surface in $\mathbb{P}^5$, and there are two types of such curves: the irreducible hyperplane sections leading to the generic module with this Hilbert function and the reducible ones, two conics with a common point, leading to a special module.

In both cases let $\mathcal{F} = Syz_1(M)$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^4}(-1) \oplus \Omega^3_{\mathbb{P}^4}(3)$. By (1.15) $\mathrm{H}^1(\mathcal{F}(*)) = \mathrm{H}^1(\mathcal{I}_S(*+4))$ and $\mathrm{H}^i(\mathcal{F}(*)) = 0$ for $i = 2, 3$. For a generic module M, $\mathcal{F}$ has a minimal free resolution of type

$$0 \leftarrow \mathcal{F} \longleftarrow \begin{array}{c} \mathcal{O} \\ \oplus \\ 15\mathcal{O}(-1) \end{array} \nwarrow \begin{array}{c} 15\mathcal{O}(-2) \\ \oplus \\ \mathcal{O}(-3) \end{array} \longleftarrow \begin{array}{c} 5\mathcal{O}(-3) \\ \oplus \\ 2\mathcal{O}(-4) \end{array} \nwarrow \mathcal{O}(-5) \leftarrow 0$$

and for M corresponding to the reducible hyperplane sections of the Veronese surface $\mathcal{F}$ has a resolution of type

$$0 \leftarrow \mathcal{F} \leftarrow \begin{array}{c} \mathcal{O} \\ \oplus \\ 15\mathcal{O}(-1) \\ \oplus \\ \mathcal{O}(-2) \end{array} \nwarrow \begin{array}{c} 16\mathcal{O}(-2) \\ \oplus \\ 3\mathcal{O}(-3) \end{array} \longleftarrow \begin{array}{c} 7\mathcal{O}(-3) \\ \oplus \\ 3\mathcal{O}(-4) \end{array} \longleftarrow \begin{array}{c} \mathcal{O}(-4) \\ \oplus \\ \mathcal{O}(-5) \end{array} \leftarrow 0$$

In both cases we have $\dim_k \mathrm{Hom}(\mathcal{E}, \mathcal{F}) = 35$ and the degeneracy locus of a generic $\varphi \in \mathrm{Hom}(\mathcal{E}, \mathcal{F})$ will be a regular surface S of degree 10, with sectional genus 9 and $\chi = 2$. The smoothness can be checked in an example with Macaulay [Mac]. More details on computational aspects can be found in the appendix A of [DES]. To identify the surfaces we use Le Barz's formula (0.11) which gives $N_6(10, 9, 2) = 3$ for the sum of the number of 6-secant lines plus the number of exceptional lines on S.

We consider first the case of the generic module. A resolution of the ideal sheaf of S is given by the mapping cone between the resolutions of $\mathcal{E}$ and $\mathcal{F}$, so it is easily seen that the homogeneous ideal of the surface is generated by quintic hypersurfaces in this case. It follows that S has no 6-secants and exactly three (-1) lines, hence it is a proper elliptic surface as in Proposition 2.1 a). This construction is the one described in [DES], Appendix B, B7.4.

In the case of the special module the mapping cone yields a minimal free resolution for $\mathcal{I}_S$ of type

$$
\begin{array}{ccccccccc}
 & & \mathcal{O}(-4) & & & & & & \\
 & & \oplus & & & & & & \\
0 \leftarrow \mathcal{I}_S \leftarrow & & 9\mathcal{O}(-5) & \nwarrow & 15\mathcal{O}(-6) & & 7\mathcal{O}(-7) & & \mathcal{O}(-8) \\
 & & \oplus & & \oplus & \longleftarrow & \oplus & \longleftarrow & \oplus & \leftarrow 0 \\
 & & \mathcal{O}(-6) & & 3\mathcal{O}(-7) & & 3\mathcal{O}(-8) & & \mathcal{O}(-9)
\end{array}
$$

In this case there is precisely one sextic generator for I_S and we guess that there is precisely one 6-secant. Indeed the three linear forms

$$\mathcal{O}(-6) \longleftarrow 3\mathcal{O}(-7)$$

in the bottom row of the resolution are independent (since those in the resolution of $\mathcal{F}$ are) and annihilate the module $I_S/(I_S)_{\leq 5}$, hence this module has support on a line L. Or one can argue as follows. This line is just the line defined by the forms in the kernel of the multiplication map

$$\mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) \longrightarrow \mathrm{Hom}(\mathrm{H}^1(\mathcal{I}_S(3)), \mathrm{H}^1(\mathcal{I}_S(4))),$$

so for all hyperplane sections H containing L we'll have $\mathrm{h}^1(\mathcal{I}_H(4)) = 1$. On the other hand the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{I}_S(2) \longrightarrow \mathcal{I}_S(3) \longrightarrow \mathcal{I}_H(3) \longrightarrow 0$$

gives $2 \leq \mathrm{h}^1(\mathcal{I}_H(3)) \leq 3$ so each hyperplane H in the net contains at least a plane π with $\mathrm{h}^1(\mathcal{I}_{\pi\cap S}(4)) = 1$. The generic hyperplane section through L is irreducible, so π cuts S only in points for such a H. Now lemma 0.41 shows that π contains a 6-secant line to the surface since if $\mathrm{h}^0(\mathcal{I}_{\pi\cap S}(2)) > 0$ the cohomology of the exact sequences

$$0 \longrightarrow \mathcal{I}_H(1) \longrightarrow \mathcal{I}_H(2) \longrightarrow \mathcal{I}_{\pi\cap S}(2) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{I}_S \longrightarrow \mathcal{I}_S(1) \longrightarrow \mathcal{I}_H(1) \longrightarrow 0$$

yields $\mathrm{h}^1(\mathcal{I}_S(1)) = \mathrm{h}^1(\mathcal{I}_H(1)) > 0$, which contradicts Severi's theorem. Therefore L is a 6-secant to S and, by Proposition 2.1, S is a non-minimal $K3$ surface with two (-1) lines and one (-1) quartic, as desired.

Remarks 2.3.
a) It is shown in [PR] *that any surface of type a) or c) in Proposition* 2.1 *can be obtained by the above construction. In particular,* (1.2) *and* (1.3) *imply that the Hilbert schemes of such surfaces are irreducible and unirational.*

b) A flat deformation of elliptic surfaces of degree 10*, sectional genus* $\pi = 9$*, to a scheme belonging to the irreducible component of the Hilbert scheme containing the* $K3$ *surfaces of type c) can be constructed by varying the hyperplane section of the Veronese surface in* $\mathbb{P}^5$*.*
c) In the above setting, all modules M *distinguish a plane* P*, assumed to be given as* $P = \{ x_0 = x_1 = 0 \}$*. In the case of the elliptic surface,* P *cuts it along a plane cubic, namely the non-exceptional part of the canonical divisor. For the general* $K3$ *surface of type c)* P *contains the unique* 6*-secant line* L *and cuts the surface only in points.* $\square$

Assume now that S is a surface of type *a*) or *c*). S can be linked in the complete intersection of the unique quartic containing it and a quintic hypersurface to an irreducible surface Y of degree 10, with sectional genus $\pi(Y) = 9$ and $\chi(Y) = 0$. The cohomology of the liaison exact sequence

$$0 \longrightarrow \mathcal{I}_{S\cup Y}(4) \longrightarrow \mathcal{I}_Y(4) \longrightarrow \mathcal{O}_S(K) \longrightarrow 0$$

gives $\mathrm{h}^0(\mathcal{I}_Y(3)) = 0$ and $\mathrm{h}^0(\mathcal{I}_Y(4)) = 2$. Therefore Y can be further linked in the complete intersection of two quartics to a surface Z of degree 6, sectional genus $\pi(Z) = 1$, $\chi(Z) = 0$.

Working out with Macaulay [Mac] the construction of S via the Eagon-Northcott complex one can figure out how the scheme Z should look like. We use in the sequel this description to give a liaison construction of a $K3$ surface of type *c*). For the elliptic case we refer to [Ra1] and [PR].

(2.4.) Liaison. Let C be a rational cubic scroll in $\mathbb{P}^4$ and let T be a smooth quadric surface cutting C along two lines in its ruling, say L_1 and L_2. Consider next a plane P passing through the directrix L of C, cutting the scroll only along this line and not contained in the hyperplane spanned by the quadric surface. Let now $Z = P \cup T \cup C$. It is easily seen that Z has the right invariants: $\deg Z = 6$ and $\pi(Z) = 1$ by formula (0.2). Also Z is locally Cohen-Macaulay and even a local complete intersection except for the points $\{p_i\} = L \cap L_i$, $i = \overline{1,2}$. We prove in the sequel that Z can be backwards linked $(4, 4)$ and $(4, 5)$ to a smooth $K3$ surface of type *c*). First a lemma.

Lemma 2.5.
a) The scheme $X = T \cup C$ *is a degenerated elliptic quintic scroll in* $\mathbb{P}^4$ *and, in particular, the homogeneous ideal* I_X *is generated by* 5 *cubics.*
b) The homogeneous ideal I_Z *is generated by* 10 *quartics and one quintic. Moreover, the quartics cut out the scheme* Z *outside* L *and the generic quartic contains* L *with multiplicity two.*

Proof. Let H be the hyperplane spanned by the quadric surface T. For the first part of the lemma we consider the residual exact sequence

$$0 \longrightarrow \mathcal{I}_C(2) \longrightarrow \mathcal{I}_X(3) \longrightarrow \mathcal{I}_{H\cap X,H}(3) \longrightarrow 0$$

of sheaves of ideals on $\mathbb{P}^4$. This sequence remains exact after taking global sections since $\mathrm{h}^1(\mathcal{I}_C(2)) = 0$. Therefore $\mathrm{h}^0(\mathcal{I}_X(3)) = 5$ and it suffices to check whether C is cut out by quadrics and $H \cap X$ by cubics. The former is clear since C is defined by the 2×2 minors of a matrix with linear entries, while $H \cap X = T \cup L$ is clearly cut out by the unions of the quadric and the hyperplanes through L. Moreover the cohomology of the above sequence yields

$$\mathrm{h}^1(\mathcal{I}_X(k)) = 0 \quad \text{for all } k, \quad \mathrm{h}^2(\mathcal{I}_X(k)) = 0 \quad \text{for} \quad k \neq 0, \quad \text{and} \quad \mathrm{h}^2(\mathcal{I}_X) = \mathrm{h}^2(\mathcal{I}_{T \cup L}) = 1.$$

Therefore Beilinson's spectral sequence gives a resolution of the form

$$0 \longrightarrow \Omega^3_{\mathbb{P}^4}(3) \longrightarrow 5\mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{I}_X(3) \longrightarrow 0$$

and hence X is a degenerated elliptic quintic scroll. Consider now the exact sequence

$$0 \longrightarrow \mathcal{I}_X(k-1) \longrightarrow \mathcal{I}_Z(k) \longrightarrow \mathcal{I}_{Z \cap H', H'}(k) \longrightarrow 0$$

where H' is a general hyperplane containing P. It remains also exact after taking global sections since $\mathrm{h}^1(\mathcal{I}_X(k)) = 0$ for all k. Now $Z \cap H' = P \cup D \cup f_1 \cup f_2$, where $D = T \cap H'$ is a smooth conic and f_1, f_2 are rulings of C, thus $\mathrm{h}^0(\mathcal{I}_{Z \cap H', H'}(k)) = \mathrm{h}^0(\mathcal{I}_{D \cup f_1 \cup f_2}(k-1))$. One easily computes $\mathrm{h}^0(\mathcal{I}_{D \cup f_1 \cup f_2}(2)) = 0$, $\mathrm{h}^0(\mathcal{I}_{D \cup f_1 \cup f_2}(3)) = \mathrm{h}^0(\mathcal{O}_{\mathbb{P}^3}(3)) - 2\mathrm{h}^0(\mathcal{O}_{\mathbb{P}^1}(3)) - \mathrm{h}^0(\mathcal{O}_{\mathbb{P}^1}(6)) = 5$ and analogously $\mathrm{h}^0(\mathcal{I}_{D \cup f_1 \cup f_2}(4)) = 16$. Moreover the homogeneous ideal $I_{D \cup f_1 \cup f_2}$ is generated by the 5 cubics and one quartic and since, by a), I_X is generated by 5 cubics the first assertion of b) follows. For the second part it is enough to observe that all cubics in $\mathrm{H}^0(\mathcal{I}_{D \cup f_1 \cup f_2}(3))$ are vanishing on L and that they cut out, scheme-theoretically in fact, $D \cup f_1 \cup f_2 \cup L$.□

As a consequence of the above lemma and remark 0.31, we obtain that Z can be linked in the complete intersection of two quartic hypersurfaces to an irreducible surface Y, with $\deg Y = 10$, $\pi(Y) = 9$, which contains and is singular along L and which is smooth outside this line. Further Y can be linked $(4, 5)$ to a surface S with the desired invariants. Namely, $\deg S = 10$, $\pi(S) = 9$ and, from the liaison exact sequences, $p_g = 1$, $q = 0$. It is easily seen that S is smooth outside L, for a general choice of the linkages. To see the behavior at the intersection with L we'll work out explicitly this liaison.

We consider the blowing-up

$$\sigma : \widetilde{\mathbb{P}^4} = \mathbb{P}(2\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)) \longrightarrow \mathbb{P}^4$$

of $\mathbb{P}^4$ along the line L, with exceptional divisor $E = \mathbb{P}(2\mathcal{O}_{\mathbb{P}^2}) = \mathbb{P}(3\mathcal{O}_L) = \mathbb{P}^2 \times L$. Let B be a divisor of E corresponding to a section of $3\mathcal{O}_L(1)$ and F corresponding to a fibre of the projection $\sigma : \mathbb{P}^2 \times L \to L$. If a hypersurface V of degree v contains the line L with multiplicity m, then its strict transform $\widetilde{V}$ will meet E along $\overline{V}$, numerically of type

$$\overline{V} \equiv mB + (v - 2m)F.$$

Let V_1 and V_2 be two general quartic hypersurfaces containing Z. By lemma 2.5 they have multiplicity two along L, thus $\overline{V_i} \equiv 2B$, $i = \overline{1, 2}$. On the other hand the strict transforms of P and T cut E along $\overline{P} \equiv (B-F)(B-F) \equiv B^2 - 2BF$ and $\overline{T} \equiv (B-F)2F \equiv 2BF$, respectively. Moreover, the strict transform of C cuts E along $\overline{C} \equiv B^2$ since the cubic scroll is linked to a plane in the complete intersection of two quadric hypersurfaces having multiplicity one along L.

It follows that, for a general $(4,4)$ liaison, the strict transform of Y on $\widetilde{\mathbb{P}^4}$ meets E in a curve equivalent to $(2B)(2B) - (B^2 + (B^2 - 2BF) + 2BF) \equiv 2B^2$. A local computation shows that the general quintic hypersurface containing Y has multiplicity one along L. Therefore, for a general choice of the $(4,5)$ liaison, the strict transform of S on $\widetilde{\mathbb{P}^4}$ will meet E in a curve equivalent to $2B(B+3F) - 2B^2 \equiv 6BF$. A Bertini argument shows now that for the general choice of the liaison, the surface S residual to Y is smooth. Moreover, since a curve of type BF is blown down on S, it follows that L is a 6-secant line to S.

To show that S is indeed a $K3$ surface we determine the one dimensional components in the intersections $S \cap T$ and $S \cap C$. The liaison exact sequences (0.24) for Y give $P \cap Y \equiv 3H_P - K_P - L \equiv 5H_P$, $T \cap Y \equiv 3H_T - K_T - [T \cap C] \equiv 5l_1 + 3l_2$, where l_1 and l_2 denote the classes of the two rulings of the quadric, and $C \cap Y \equiv 3H_C - K_C - [C \cap (P \cup T)] \equiv 3(C_0 + 2f) - (-2C_0 - 3f) - C_0 - 2f \equiv 4C_0 + 7f$, with C_0 denoting the numerical class of the directrix L on the scroll C and f the class of a ruling. The one-dimensional components of the intersection scheme $S \cap Z$ are the residuals (in term of conductor ideals) of the above curves in the complete intersection of Z with the quintic hypersurface used in the liaison of Y with S. Therefore, for a general choice of the liaison, P cuts S only in points, C cuts S along a scheme whose one-dimensional part K_1 is equivalent to $5H_C - [C \cap Y] \equiv C_0 + 3f$ and T cuts S along a curve K_2 equivalent to $5H_T - [T \cap C] \equiv 2l_2$, plus a zero dimensional scheme. On the other hand, a similar computation on S shows that the scheme $K_1 \cup K_2$ is exactly the canonical divisor of S. Now the liaison exact sequence

$$0 \longrightarrow \mathcal{I}_{Y \cup Z}(5) \longrightarrow \mathcal{I}_{Y \cup P \cup C}(5) \longrightarrow \mathcal{O}_T \longrightarrow 0$$

remains also exact after taking global sections. It follows that the quintics in $\mathrm{H}^0(\mathcal{I}_Y(5))$ cut on T a linear system whose fixed part is exactly $T \cap Y$. Therefore, for a general choice of the $(4,5)$ liaison, the curve $K_2 \subset Z \cap S$ is reduced and hence it is the union of two skew lines, say E_2 and E_3, in the ruling of T which contains L_1 and L_2. Eventually, the adjunction formula on S yields $E_i^2 + KE_i = 2E_i^2 = 2p_a(E_i) - 2 = -2$, $i = \overline{2,3}$ and thus E_2 and E_3 are exceptional lines on S. By (2.1), it follows that $E_1 := K_1$ is a (-1) quartic on S and hence S is, as claimed, a $K3$ surface of type c); i.e., S is embedded by a linear system of type

$$H = H_{min} - 4E_1 - E_2 - E_3.$$

Concerning liaison we want to add the following

Remark 2.6. *Z is the unique minimal scheme in the even liaison class of S. Therefore, by the general results in* [BBM] *and* [MDP], *S can be viewed as obtained from Z via a basic double link and a flat deformation.*

Proof. The arguments used in the proof of Lemma 2.5 give also $p_g(Z) = 0$, $q(Z) = 1$ and, moreover, that $e(Z){:=}\max\{\, t \mid h^2(\mathcal{O}_Z(t)) \neq 0 \,\} = -2$. Therefore the above claims follow from (0.32) and the general description of a liaison class in [BBM].

3. Smooth surfaces of degree 11

We construct in this chapter several families of smooth surfaces of degree 11 in $\mathbb{P}^4$, providing along also an incomplete and rough attempt of numerical classification. In a certain sense, the classification is to be considered only as a guideline for where one should look to in order to find the desired examples.

We start by recalling the basic relations between the invariants of a smooth surface S of degree 11 in $\mathbb{P}^4$. The double point formula (0.6) takes now the form

$$K^2 = 6\chi - 5\pi + 38$$

and Severi's theorem (0.7) and Riemann-Roch give

$$\pi = \chi + 7 + \mathrm{h}^1(\mathcal{O}_S(H)) - \mathrm{h}^0(\mathcal{O}_S(K-H)).$$

In addition to these relations Theorem 1.6 says that S is contained in a hyperquadric if and only if $\pi = 1 + \left[\frac{11(11-4)}{4}\right] = 20$, and in this case S is linked to a plane in the complete intersection of the hyperquadric with a degree 6 hypersurface.

We assume therefore in the sequel that S doesn't lie on a hyperquadric. A theorem of Gruson and Peskine [GP1], [Au1] provides then $\pi \leq 15$, and furthermore, when equality holds, that S is linked to a plane in the complete intersection of a cubic and a quartic hypersurfaces. Therefore we are left to study only the cases when $\pi \leq 14$. We look now for a lower bound for the sectional genus and we start with the smallest values of π.

Proposition 3.1. *If S is a smooth surface of degree 11 in $\mathbb{P}^4$, then $\pi \geq 8$.*

Proof. When $\pi \leq 7$ then $HK \leq 1$, so the index theorem implies $K^2 \leq \frac{(HK)^2}{11} < 1$. On the other hand the double point formula gives

$$\chi = \frac{K^2 + 5\pi - 38}{6} < 0$$

whence S is birationally ruled, which in turn means that $K^2 \leq 8\chi$. But this implies $2\chi \geq 38 - 5\pi \geq 3$ which contradicts the above inequality.$\square$.

I. Surfaces with $d = 11$, $\pi = 8$

The non-existence of smooth, sectionally non-special surfaces with these invariants was shown in [MR]. We'll recall their results along the proof of the following proposition which mainly collects the data in [MR]:

Proposition 3.2. *A smooth surface $S \subset \mathbb{P}^4$ of degree 11, with sectional genus $\pi = 8$ is necessarily a non-minimal abelian surface with one (-1) line and one (-1) conic.*

Proof. In this case $HK = 3$ so, as above, the double point formula and the index theorem give $\chi \leq 0$.

Strict inequality means that S is birationally ruled, whence $K^2 \leq 8\chi$, which leaves as only possibility $\chi = -1$. Therefore $K^2 = -8$ and S is in fact geometrically ruled over a curve of genus 2. But this is impossible by the results in [HR]. It is also possible to see this directly by using adjunction theory since, by Theorem 0.13, $S_1 = \varphi_{H+K}(S) \subset \mathbb{P}^5$ would be a smooth surface with invariants $d_1 = 9$, $H_1K_1 = -5$, $\pi_1 = 3 > q = 2$, $K_1^2 = -8$, and so adjunction again would give $0 \leq (H_1 + K_1)^2 = 9 - 10 - 8 = -9$, a contradiction.

Thus $\chi = 0$, $K^2 = -2$ and, in particular, S is not of general type. The adjunction mapping is birational and the image $S_1 = \varphi_{H_1+K_1}(S) \subset \mathbb{P}^6$ is a smooth surface of degree $d_1 = 15$ and sectional genus $\pi_1 = 9$. Also $H_1K_1 = 1$ and $K_1^2 = -2 + a$, where $a \geq 0$ is the number of (-1) lines on S. In general we'll denote by S_n the image of the n-th iterated adjunction mapping and, in case it is a surface, by H_n the hyperplane section, by K_n the canonical divisor, by π_n the sectional genus and by a_n the number of (-1) lines on S_{n-1}, or equivalently the number of (-1) rational curves of degree n on S. In particular we let $S_0 = S$ and $a_1 = a$.

If $p_g > 0$, then K_1 is a line and $K_1^2 = -1$ since $(H_1 + K_1)K_1 \geq 0$ and S is not ruled. In this case S is a blown-up abelian surface with one (-1) line and one (-1) conic. The minimal model is embedded by the second adjunction as a surface of degree 16 and sectional genus 9 in $\mathbb{P}^7$.

We assume from now on $p_g = 0$. If $a = 0$, then $(H_1 + K_1)K_1 < 0$ so S is birationally ruled. If $K_2^2 = -2$, then $S_4 = \varphi_{H_3+K_3}(S_3) \subset \mathbb{P}^3$ would be a ruled surface with $d_4 = a_3 + 3$ and $\pi_4 = a_3$, for $a_3 \in \{0, 1, 2\}$, which is absurd. If $K_2^2 = -1$, then $S_4 \subset \mathbb{P}^4$ would be a surface with $d_4 = 7 + a_3$ and $\pi_4 = 3 + a_3$, with $a_3 \in \{0, 1\}$, which is impossible by the classification of such surfaces in [Io1],[Io2],[Ok3],[Ok4]. Finally, if $K_2^2 = 0$, then S_2 is minimal and the adjunction process would embed $S_5 \subset \mathbb{P}^4$ as a smooth surface with $d_5 = 9$ and $\pi_5 = 5$, which is impossible by the classification in [AR].

If $a = 1$, then the adjoint mapping for S_1 is again birational on a surface $S_2 \subset \mathbb{P}^7$, having invariants $d_2 = 16$, $\pi_2 = 9$, $H_2K_2 = 0$ and $K_2^2 = -1 + a_2$, where $a_2 \in \{0, 1\}$ by Hodge index. For $a_2 = 0$, $S_5 \subset \mathbb{P}^{4+a_3}$ is a smooth surface with $d_5 = 7 + 4a_3 + a_4$ and $\pi_5 = 3 + 3a_3 + a_4$. Thus if $a_3 = 0$, it is a smooth surface of degree 7 or 8 and sectional genus 3 or 4 respectively, in $\mathbb{P}^4$, which is as above a contradiction, while if $a_3 = 1$, then $S_6 \subset \mathbb{P}^4$ would have invariants $d_6 = 9$, $\pi_6 = 5$, $K_6^2 = 0$, which is absurd by [AR]. Assume now $a_2 = 1$. In this case S_2 is minimal bielliptic because the adjunction process yields surfaces $S_n \subset \mathbb{P}^7$ with the same numerical invariants: $d_n = 16$, $H_nK_n = 0$, $\pi_n = 9$, for all $n \geq 3$. In [MR] it is shown that such smooth bielliptic surfaces $S_2 \subset \mathbb{P}^7$ exist, while the corresponding surfaces in $\mathbb{P}^4$ are necessarily singular.

If $a = 2$, then $K_1^2 = 0$ and S_1 is already minimal. It is not birationally ruled since an argument as in (0.18) shows that the adjunction doesn't stop. Since K_1 is not numerically trivial it follows that S would be a blown-up proper elliptic surface with two (-1) lines.

Using Reider's criterion, in [MR] it is shown that smooth minimal proper elliptic surfaces $S_1 \subset \mathbb{P}^6$ with $d_1 = 15$, $\pi_1 = 6$, $p_g = 0$ and $q = 1$ do not exist.□

Remark 3.3. *We have not been able to find either examples nor to prove the non-existence of smooth abelian surfaces as in the above proposition. The author believes they don't exist.*

II. Surfaces with $d = 11$, $\pi = 9$

Adjunction and the double point formula give $HK = 5$, $K^2 = 6\chi - 7$, thus $\chi \leq 1$ by Hodge index. On another side, $0 \leq \chi \leq 1$ by (0.20) if S is birationally ruled or rational, while (0.16) yields $\chi = 1$ when $\kappa(S) \geq 0$. Again, as in the previous chapter, we have not been able to construct examples of smooth surfaces with these invariants. In terms of numerical invariants we mention the following remarks:

Proposition 3.4. *There are no smooth surfaces $S \subset \mathbb{P}^4$ with $d = 11$, $\pi = 9$ and $\chi = 0$.*

Proof. As argued above, if such a surface exists, then it is birationally ruled over an elliptic curve. In particular $\mathrm{h}^1(\mathcal{O}_S) = 1$ and $p_g = 0$. On another side, Riemann-Roch and Severi's theorem yield for the speciality $\mathrm{h}^1(\mathcal{O}_S(1)) = 2$. It follows that the variety in $\check{\mathbb{P}}^4$ parametrizing hyperplane sections H for which $\mathrm{h}^1(\mathcal{O}_H(1)) = 2$ is a plane, so there is a line $E \subset \mathbb{P}^4$ which is the base locus of the net of hyperplanes for which $\mathrm{h}^1(\mathcal{O}_H(1)) = 2$. To prove the proposition we'll use the following fact [ACGH, p.198, ex. E-1]:
Let C be a smooth curve of genus g and let L be a line bundle of degree d. For $r = \mathrm{h}^0(L) - 1$ the inequality

$$d \geq g - d + 2r + \mathrm{h}^1(L^{\otimes 2}) \qquad (*)$$

holds.
If E doesn't lie on S, then the general H in the net is smooth and applying $(*)$ for $C = H$ and $L = \mathcal{O}_H(K)$ we obtain

$$5 \geq 9 - 5 + 2 + \mathrm{h}^1(\mathcal{O}_H(2K)),$$

which is a contradiction. Therefore E lies on S and is contained in a fibre of the ruling, since otherwise it would dominate the elliptic base of the minimal model of S. If the general section $C \in |H - E|$ is not integral, then, by Bertini's theorem, $|H - E|$ is composed with a pencil of plane curves. It has no base points, since a basepoint would be a singular point on S, thus $(H - E)^2 = 0$ and hence $E^2 = -9$, $p_a(C) = E^2 + 9 = 0$, which is impossible for a union of plane quintic curves, or conics. Therefore the general $C \in |H - E|$ is a smooth and irreducible curve of degree 10 and genus $g(C) = E^2 + 9$. We look now for the degrees of the exceptional curves on S. Let S_1 denote the image of S under the adjunction map and S_2 the image of S_1 under the map defined by $|H_1 + K_1|$. We compute the following invariants:

$S \subset \mathbb{P}^4$	$H^2 = 11$	$HK = 5$	$K^2 = -7$	$\pi = 9$
$S_1 \subset \mathbb{P}^7$	$H_1^2 = 14$	$H_1K_1 = -2$	$K_1^2 = -7 + a$	$\pi_1 = 7$
$S_2 \subset \mathbb{P}^5$	$H_2^2 = 3 + a$	$H_2K_2 = -9 + a$	$K_2^2 = -7 + a + b$	$\pi_2 = -2 + a$,

where a and b are non-negative numbers. If $a \geq 4$, then $\pi_2 > q$ and theorem 0.13 implies that $(H_2 + K_2)^2 = -22 + 4a + b \geq 0$, whence in fact $a \geq 5$. If $a = 5$, then $b = 2$ and S_2 would be via adjunction a conic bundle over $\mathbb{P}^1$, which is absurd for $q = 1$; if $a = 6$, then the adjunction map on S_2 would have degree $(b + 2)$ over $\mathbb{P}^2$, which is impossible by (0.13), while if $a = 7$, then $b = 0$ and the image of S_2 under the adjunction map would be a surface of degree 6 in $\mathbb{P}^3$, which is again absurd. It follows that $a = 3$ and $\pi_2 = 1$, hence

S_2 is an elliptic scroll of degree 6 in $\mathbb{P}^5$. In particular, there are only exceptional lines and exceptional conics on S, hence E is either a (-1) line, or $(H+K)E = 1$ and E is a (-2) line. In the former case, the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_E(E) \longrightarrow \mathcal{O}_H(H) \longrightarrow \mathcal{O}_C(H) \longrightarrow 0$$

yields $\mathrm{h}^1(\mathcal{O}_C(H)) = \mathrm{h}^1(\mathcal{O}_H(H)) = 2$. Now again the inequality $(*)$, applied this time for C and $L = \mathcal{O}_C(K-E)$, leads to a contradiction. A similar argument rules out the case $E^2 = -2$, and thus surfaces with the above invariants do not exist.$\square$

Proposition 3.5. *A smooth surface with invariants $d = 11$, $\pi = 9$ and $\chi = 1$ is either*
a) rational, or
b) a non-minimal Enriques surface with one (-1) quintic, or
c) a non-minimal regular proper elliptic surface with one (-1) line, or one (-1) conic.

Proof. The first two cases are clear, while regularity follows from the same argument we've been using in proposition 3.4. Assume now $\kappa(S) \geq 1$ and let S_1 denote the image of S through the adjunction mapping. Let H_1 denote the hyperplane divisor and K_1 the canonical class on S_1. We obtain the following list of invariants:

$S \subset \mathbb{P}^4$	$H^2 = 11$	$HK = 5$	$K^2 = -1$	$\pi = 9$
$S_1 \subset \mathbb{P}^8$	$H_1^2 = 20$	$H_1K_1 = 4$	$K_1^2 = -1 + a$	$\pi_1 = 13$
$S_2 \subset \mathbb{P}^{12}$	$H_2^2 = 27 + a$	$H_2K_2 = 3 + a$	$K_2^2 = -1 + a + b$	$\pi_2 = 16 + a,$

where a and b are the number of the (-1)-lines and (-1)-conics on S respectively. We observe that $(H_1K_1)^2 = 16 < 20 = H_1^2$, thus S is not of general type by lemma 0.18. On another side $\mathrm{h}^0(\mathcal{O}_S(2K)) \neq 0$ by Castelnuovo's rationality criterion, hence $HK_{min} \geq 2$. If $HK_{min} = 2$, we have $\mathrm{h}^0(\mathcal{O}_S(2K)) = 1$, since an elliptic curve must have degree at least 3, but then an easy argument using Kodaira's formula for the canonical bundle of an elliptic fibration [BPV] shows in fact that this case cannot occur. Therefore $HK_{min} \geq 3$ and S is a non-minimal proper elliptic surface with only one exceptional curve of degree less or equal to 2. $\square$

III. Surfaces with $d = 11$, $\pi = 10$

The double point formula gives $K^2 = 6\chi - 12$ and Hodge index implies $K^2 \leq 4$, thus $\chi \leq 2$. On the other hand, lemma 0.20 yields $\chi \geq 0$ when $\kappa(S) = -\infty$, while the pseudo-effectiveness of K gives $\chi \geq 1$ for $\kappa(S) \geq 0$.

Proposition 3.6. *A smooth surface $S \subset \mathbb{P}^4$ with $d = 11$, $\pi = 10$ and $\chi = 1$ is either a rational surface or a blown-up Enriques surface, embedded by*

$$H = H_{min} - 2E_1 - \sum_{i=2}^{6} E_i.$$

Proof. Let S_1 be the image of S through the adjunction morphism. From (0.13) we obtain the following invariants

$S_1 \subset \mathbb{P}^9 \qquad H_1^2 = 19 \qquad H_1K_1 = 1 \qquad K_1^2 = -6 + a \qquad \pi_1 = 11,$

where a is the number of (-1) lines on S. Furthermore $p_g = 0$. Otherwise, since S_1 is not a scroll and $(H_1 + K_1)K_1 \geq 0$ by (0.16), a divisor in $|K_1|$ would be an exceptional line on S_1, and thus the image of S_1 through adjunction would be a surface with trivial canonical class and $\chi = 1$, absurd. Thus S is regular, and either $p_2 = 0$ and S is rational by Castelnuovo's theorem, or $\mathrm{h}^0(\mathcal{O}_S(2K_1)) > 0$ and a divisor in $|2K_1|$ is twice an exceptional line. In the last case, the image of S_1 through the adjunction map is a minimal Enriques surface $S_2 \subset \mathbb{P}^{10}$, with $d_2 = 20$ and $\pi_2 = 11$, whereas S is embedded by

$$H = H_2 - 2E_1 - \sum_{i=2}^{6} E_i. \square$$

(3.7.) Determinantal construction. An example of an Enriques surface S with the above invariants and minimal cohomology was constructed in [DES]. Namely, assuming S has the Beilinson cohomology table

i

	2			
		1	5	5

p

$\mathrm{h}^i(\mathcal{I}_S(p))$

the authors construct the surface as the degeneracy locus of a general morphism $\varphi \in \mathrm{Hom}(2\Omega^3(3), \mathcal{G})$, with $\mathcal{G} = \ker(10\mathcal{O} \xrightarrow{\psi} \mathcal{O}(2))$ and ψ a quite special morphism. Namely, let

$E \subset \mathbb{P}^4$ be an elliptic normal curve, let τ be a non-trivial 2-torsion point on E and denote by $\overline{p, p+\tau}$ the line in $\mathbb{P}^4$ spanned by p and $p+\tau$. Then (cf. [BHM])

$$Q = \bigcup_{p \in E} \overline{p, p+\tau} \subset \mathbb{P}^4$$

is an elliptic quintic scroll over the elliptic curve E/τ, which contains E as a 2-section. Corresponding to the other two unramified 2:1 covers of E/τ, Q contains two further elliptic normal curves E' and E'' of which it is a 2-translation scroll. The three elliptic curves are mutually disjoint. In this setting, the authors of [DES] take ψ to be defined by the ten hyperquadrics which generate $I_{E'} + I_{E''}$. As checked in examples, the rulings of Q are 6-secant lines to the constructed surface S.

(3.8.) Liaison construction. As a completion of the determinantal construction in [DES], we provide in the sequel an equivalent liaison construction for this family of Enriques surfaces.

Let $p_1, \ldots, p_5$ be five distinct points on the elliptic normal curve E such that $p_1 + p_2 + \ldots + p_5 = \tau$, where τ is a non-trivial 2-torsion point on E, and let $\pi \subset \mathbb{P}^4$ be a general plane. There are four independent hyperquadrics containing π and the points p_i, and a Bertini argument shows that we can link π in the complete intersection of two general hyperquadrics to a smooth rational cubic scroll T, which meets the elliptic curve E only in the points p_i, $i = \overline{1,5}$.

Lemma 3.9. *The homogeneous ideal $I_{T \cup E}$ is generated by 3 cubic and 5 quartic hypersurfaces.*

Proof. Consider the cohomology associated to the residual intersection exact sequences

$$0 \longrightarrow \mathcal{I}_E(k-2) \xrightarrow{f} \mathcal{I}_{E \cup T}(k) \longrightarrow \mathcal{I}_{T \cup Z, V}(k) \longrightarrow 0, \qquad k \in \mathbb{Z},$$

where f is the equation of a general hyperquadric V containing T and $Z = V \cap E \setminus T \cap E$ is a reduced scheme of length 5, disjoint of T. Now $\mathrm{h}^1(\mathcal{I}_E(m)) = 0$, for all $m \in \mathbb{Z}$ (e.g., cf. [Hu]), $\mathrm{h}^0(\mathcal{I}_{T,V}(3)) = 8$, $\mathrm{h}^0(\mathcal{I}_{T,V}(4)) = 20$, while $\mathcal{I}_E(2)$ and $\mathcal{I}_{T \cup Z, V}(4)$ are clearly generated by global sections, so the lemma follows.□

We remark that a cubic hypersurface containing $T \cup E$ is obviously irreducible, and by a result of Aure [Au1, Lemma 2.1.6, Lemma 3.1.19], cannot be a cone or have a double plane. Moreover, it has only isolated singularities since its general hyperplane section is a Del Pezzo surface (it cannot be a cone over a smooth plane cubic curve because then the corresponding hyperplane section of T would dominate the base of the cone).

T can be linked in the complete intersection of two cubic hypersurfaces in $\mathrm{H}^0(\mathcal{I}_{E \cup T}(3))$ to a surface $B \subset \mathbb{P}^4$ of degree 6 and sectional genus 3. For a general choice of the cubics B is smooth, thus it is a Bordiga surface (cf. [Ok2], [Io1]).

Lemma 3.10. *The 3 cubic hypersurfaces in* $\mathrm{H}^0(\mathcal{I}_{E\cup T}(3))$ *intersect along* $T \cup E$ *plus the union of five skew lines on* B*, say* $E_1, E_2, \ldots, E_5$*, which are secants to the elliptic curve* E*.*

Proof. B is a rational surface in $\mathbb{P}^4$, embedded by the linear system

$$|H_B| = |4l - \sum_{i=1}^{10} E_i|$$

and, by construction, it contains E and it intersects the rational scroll T along a curve $G \equiv H_B - K_B$, of degree 8 and arithmetic genus 5. In terms of the embedding we have

$$G \equiv 7l - \sum_{i=1}^{10} 2E_i,$$

and we may write

$$E \equiv al - \sum_{i=1}^{10} a_i E_i,$$

where $a \in \mathbb{N}^*$ and $a_i \in \mathbb{N}$, for $i = \overline{1,10}$. Now $G \cap E = T \cap E = \{p_1, \ldots, p_5\}$, so

$$GE = 7a - 2\sum_{i=1}^{10} a_i = 5,$$

while

$$H_B E = 4a - \sum_{i=1}^{10} a_i = 5,$$

and

$$E^2 + K_B E = a^2 - 3a + \sum_{i=1}^{10} a_i - \sum_{i=1}^{10} a_i^2 = 0.$$

We obtain $a = 5$, $\sum_{i=1}^{10} a_i = 15$ and

$$\sum_{i=1}^{10} \frac{a_i(a_i - 1)}{2} = 5.$$

On the other hand, $EE_i = a_i \leq 2$ since E has no trisecants, so the last relation implies that exactly five of the a_i's, say $a_1, a_2, \ldots, a_5$, are equal to 2, while the rest are equal to 1. Therefore

$$E \equiv 5l - \sum_{i=1}^{5} 2E_i - \sum_{j=6}^{10} E_j,$$

and so the residual of $G+E$ in the intersection of B with a general cubic hypersurface of $\mathrm{H}^0(\mathcal{I}_{E\cup T}(3))$ is the unique curve in the linear system

$$|3H_B - E - G| = |\sum_{i=1}^{5} E_i|. \qquad \square$$

Lemma 3.11. *For general choices, B intersects the elliptic quintic scroll Q along E and a zero-dimensional smooth scheme Σ of length 5, outside the elliptic curve.*

Proof. T meets Q in 15 points, five of which, by construction, lie on E. Let now $\rho : Q \to E/\tau$ denote the morphism which defines the ruling of Q and let C_0 denote the class of a section of Q with minimal self-intersection $C_0^2 = 1$. Then $E \equiv 2C_0 - \alpha F$, where αF is the pullback by ρ of a divisor of degree one on E (cf. [HV]), while $H_Q \equiv C_0 + \beta F$, with βF the pullback of a divisor of degree two. Thus, residual to E in the complete intersection of Q with a cubic hypersurface containing E, there is a curve numerically equivalent to $C_0 + \gamma F$, with γF the pullback of a divisor of degree 7 on E, and the lemma follows since $(C_0 + \gamma F)^2 = 14 + 1 = 15$.$\square$

Lemma 3.12. *There exists a unique quintic hypersurface V containing both the elliptic scroll Q and the Bordiga surface B.*

Proof. We use the residual exact sequences

$$0 \longrightarrow \mathcal{I}_Q(2) \xrightarrow{f} \mathcal{I}_{Q\cup B}(5) \longrightarrow \mathcal{I}_{B\cup M,V}(5) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{I}_B(2) \xrightarrow{g} \mathcal{I}_{B\cup M}(5) \longrightarrow \mathcal{I}_{E\cup M\cup L,W}(5) \longrightarrow 0,$$

where $V = \{f = 0\}$ is a general cubic hypersurface containing $B \cup T$, $W = \{g = 0\}$ is a general cubic hypersurface containing the elliptic scroll Q, while $M \sim 3H_Q - E$ and $L \equiv 3H_B - E$ are the corresponding residual intersections. Now $\mathrm{h}^1(\mathcal{I}_Q(2)) = \mathrm{h}^1(\mathcal{I}_B(2)) = \mathrm{h}^0(\mathcal{I}_Q(2)) = \mathrm{h}^0(\mathcal{I}_B(2)) = 0$, and it is easily checked that $\mathrm{h}^0(\mathcal{I}_{E\cup L\cup M}(5)) = 2\cdot 15 + 1 = 31$, when the points $p_1, \ldots, p_5$ are chosen such that the sum $p_1 + p_2 + \ldots + p_5$ is a non-trivial 2-torsion point on E.$\square$

We link now B in the complete intersection of V and a general cubic hypersurface in $\mathrm{H}^0(\mathcal{I}_B(3))$ to a surface $Z \subset \mathbb{P}^4$ of degree 9, sectional genus 9. For general choices, the surface Z is smooth and meets the elliptic quintic scroll along a smooth curve M of degree 10 and genus 1 in the linear system $|3H_Q - E|$, which passes through the five points of the scheme Σ. Therefore the scheme theoretic union $Y = Q \cup Z$ is a local complete intersection scheme of degree 14 and sectional arithmetic genus 19.

Lemma 3.13. $\mathrm{h}^0(\mathcal{I}_Y(4)) = 0$ *and* $\mathrm{h}^0(\mathcal{I}_Y(5)) = 2$.

Proof. One argues as in the proof of lemma 3.12.$\square$

Therefore, we can link the configuration Y in the complete intersection of the above two quintic hypersurfaces to a surface $S \subset \mathbb{P}^4$ of degree 11 and sectional genus 10. Moreover, as we've checked in an example, the surface S is smooth for a general choice of the initial data.

Lemma 3.14. *The surface S has infinitely many 6-secant lines, so Le Barz's formula doesn't apply. Namely, the 6-secants to S are precisely the lines in the ruling of Q.*

Proof. By (0.24), the scheme $S \cup Z$ intersects the scroll Q along a curve numerically equivalent to $5H_Q - K_Q$, whence S meets Q along a curve $D \equiv 5H_Q - K_Q - M \equiv 6C_0 + \delta F$, where δ is the pullback of a divisor of degree 2 on E/τ. Since $DF = 6$, the rulings of the scroll Q are 6-secant lines to S. There are no other 6-secants to S because Z is a minimal surface of general type, which intersects S along a curve $D' \equiv 5H_Z - K_Z - M$, so for a line L on Z one would have $D'L = (5H_Z - K_Z - M)L \leq 5H_Z L = 5$. □

Lemma 3.15. *$E_1, E_2, \ldots, E_5$ are the exceptional lines of S.*

Proof. By lemma 3.10, each line E_i, $i = \overline{1,5}$, is a secant of E, and thus also of the scroll Q. On another side, B intersects Z along a curve $G' \equiv 3H_B - K_B \equiv 15l - \sum_{i=1}^{10} 4E_i$, thus a line E_i is also a 4-secant to the surface Z in points lying outside the scroll. Altogether, each E_i, for $i = \overline{1,5}$, is a 6-secant to the configuration Y, and thus it lies on any quintic hypersurface containing Y. In particular, the lines E_i, $i = \overline{1,5}$, lie on S and it remains to show they are exceptional. The Bordiga surface B is defined by the maximal minors of a 4×3 matrix with linear entries, thus Z and also $S \cup Q$, being linked with B, are projectively Cohen-Macaulay schemes. In particular the ideal sheaf of $S \cup Q$ has a minimal free resolution of type

$$0 \leftarrow \mathcal{I}_{S \cup Q} \longleftarrow 5\mathcal{O}(-5) \nwarrow \begin{matrix} 3\mathcal{O}(-6) \\ \oplus \\ \mathcal{O}(-7) \end{matrix} \leftarrow 0$$

Dualizing, we see that the minors of the 5×3 submatrix with linear entries in the above resolution define the zero-set of a section in $\mathrm{H}^0(\omega_{S \cup Q}(-2))$. It follows that these minors cut out on S an effective divisor in the class $|K + D - 2H|$, which contains $\sum_{i=1}^5 E_i$ and thus must be equal to, by degree reasons. Now $DE_i = 2$, because the lines E_i are only simple secants to the scroll Q, and since

$$DE_i = (2H - K + \sum_{i=1}^{5} E_i)E_i = 2HE_i - 2p_a(E_i) + 2 + 2E_i^2,$$

we obtain $E_i^2 = -1$, for $i = \overline{1,5}$, and the lemma is proved.□

Corollary 3.16. *The surface S is a non-minimal Enriques surface, embedded by*

$$H = H_{min} - 2E_0 - \sum_{i=1}^{5} E_i.$$

Proof. In view of proposition 3.6, it is enough to show that $\mathrm{h}^0(\mathcal{O}_S(2K)) > 0$, or equivalently that $\mathrm{h}^0(\mathcal{O}_S(2K - 2\sum_{i=1}^5 E_i)) = \mathrm{h}^0(\mathcal{O}_S(4H - 2D)) \neq 0$. If D' denotes as in (3.14) the intersection curve of S and Z, then the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_S(K-D-H) \longrightarrow \mathcal{O}_S(4H-2D) \longrightarrow \mathcal{O}_{D'}(4H-2D) \longrightarrow 0$$

together with the vanishing of $\mathrm{h}^1(\mathcal{O}_S(K-D-H)) = \mathrm{h}^1(\mathcal{O}_S(D+H)) = 0$, since $D+H$ is ample, imply that it is enough to check that $\mathrm{h}^0(\mathcal{O}_{D'}(4H-2D)) \neq 0$. But, by construction $\mathcal{O}_{D'}(D) = \mathcal{O}_{D'}(M)$, so we need in fact to show that $\mathrm{h}^0(\mathcal{O}_{D'}(4H-2M)) \neq 0$, and we check this on Z, where since $D'+M \sim 5H_Z - K_Z$ this is equivalent to showing that $\mathrm{h}^1(\mathcal{O}_{D'}(H_Z+M)) \neq 0$. If $N \sim 3H_Z - K_Z$ denotes the intersection of Z and B, then lemma 3.11 and the remarks after lemma 3.12 imply on Z that $MN = ME + 5 = 20$, where the last intersection number is computed on the scroll Q. Thus $M^2 = -MK_Z = -10$ and a similar argument shows in fact that $\mathcal{O}_M(H_Z - K_Z) = \mathcal{O}_M$. In particular $\mathcal{O}_M(H_Z+M) = \mathcal{O}_M(K_Z+M+H_Z-K_Z) = \omega_M = \mathcal{O}_M$ and thus $\mathrm{h}^1(\mathcal{O}_Z(H_Z+M)) = \mathrm{h}^1(\mathcal{O}_M(H_Z+M)) = \mathrm{h}^1(\mathcal{O}_M) = 1$ since Z is projectively Cohen-Macaulay. It follows from the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(M+H_Z-D') \longrightarrow \mathcal{O}_Z(M+H_Z) \longrightarrow \mathcal{O}_{D'}(M+H_Z) \longrightarrow 0$$

that all we need to check is that the map induced by the multiplication with the equation of the divisor D'

$$\mathrm{H}^1(\mathcal{O}_Z(M+H_Z-D')) \xrightarrow{\quad D' \quad} \mathrm{H}^1(\mathcal{O}_Z(M+H_Z)) = \mathbb{C}$$

is trivial. This is a consequence of the commutativity of the diagram

$$\begin{array}{ccccc}
\mathrm{H}^1(\mathcal{O}_Z(M+H_Z-D')) & \xrightarrow{D'} & \mathrm{H}^1(\mathcal{O}_Z(M+H_Z)) & \xrightarrow{K_Z+M} & \mathrm{H}^1(\mathcal{O}_Z(2M+H_Z+K_Z)) \\
\big\downarrow{\scriptstyle K_Z+D'} & & & & \big\downarrow{\scriptstyle \cong} \\
\mathrm{H}^1(\mathcal{O}_Z(M+H_Z+K_Z)) & \xrightarrow{\text{restr.}} & \mathrm{H}^1(\mathcal{O}_M(M+H_Z+K_Z)) & \xrightarrow{0} & \mathrm{H}^1(\mathcal{O}_M)
\end{array}$$

□

As a remark to the above liaison construction, we sketch in the sequel an alternative description of the H^1-module of these Enriques surfaces. We need first some extra cohomological information:

Lemma 3.17. $\mathrm{h}^0(\mathcal{O}_S(2H - \sum_{i=1}^5 E_i)) = 1$.

Proof. Since, by (1.6), $\mathrm{h}^0(\mathcal{I}_S(2)) = 0$, it is enough to show on the Bordiga surface B that $\mathrm{h}^0(\mathcal{O}_B(2H_B - \sum_{i=1}^5 E_i)) = 1$. By lemma 3.10 this is equivalent to showing that $\mathrm{h}^0(\mathcal{O}_B(E-K_B)) = 1$. Now we observe that $\mathrm{h}^0(\mathcal{O}_B(-K_B)) = 0$, since otherwise $E \equiv (2l - \sum_{i=1}^5 E_i) + (3l - \sum_{i=1}^{10} E_i)$ would be reducible. Hence, by Riemann-Roch, $\mathrm{h}^1(\mathcal{O}_B(-K_B)) = 0$ and the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_B(-K_B) \longrightarrow \mathcal{O}_B(E-K_B) \longrightarrow \mathcal{O}_E(E-K_B) \longrightarrow 0$$

yields $\mathrm{h}^0(\mathcal{O}_B(E-K_B)) = \mathrm{h}^0(\mathcal{O}_E(E-K_B))$. Let now as in (3.10) $G \equiv H_B - K_B$ be the intersection of T and B. Then, by construction $G \cap E = \{p_1, p_2, \ldots, p_5\}$, while $\mathcal{O}_E(K_B +$

$E) = \omega_E = \mathcal{O}_E$ by adjunction, and $\mathcal{O}_E(K_B+H_B) = \mathcal{O}_E(2H_B-G) = \mathcal{O}_E(p_1+p_2+\ldots+p_5)$ since $2(p_1+p_2+\ldots+p_5) = 2\tau = 0$ in the group structure of E. Therefore $\mathcal{O}_E(E-K_B) = \mathcal{O}_E(G+(K_B+E)-(K_B+H_B)) = \mathcal{O}_E$ and the claim of the lemma follows.□

Lemma 3.18. $\mathrm{h}^1(\mathcal{O}_S(1)) = 2$ *and* $\mathrm{h}^1(\mathcal{O}_S(k)) = 0$ *for* $k \geq 2$.

Proof. The first assertion follows from Riemann-Roch and Severi's theorem. To show that $\mathrm{h}^1(\mathcal{O}_S(2)) = 0$ we consider the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_S(2H - \sum_{i=1}^{5} E_i) \longrightarrow \mathcal{O}_S(2H) \longrightarrow \mathcal{O}_{\sum_{i=1}^{5} E_i}(2H) \longrightarrow 0.$$

Now $\mathrm{h}^1(\mathcal{O}_{\sum_{i=1}^{5} E_i}(2H)) = \sum_{i=1}^{5} \mathrm{h}^1(\mathcal{O}_{\mathbb{P}^1}(2)) = 0$, $\chi(\mathcal{O}_S(2H - \sum_{i=1}^{5} E_i)) = 1$ by Riemann-Roch, while $\mathrm{h}^2(\mathcal{O}_S(2H - \sum_{i=1}^{5} E_i)) = \mathrm{h}^0(\mathcal{O}_S(K + \sum_{i=1}^{5} E_i - 2H)) = 0$ because $H(K + \sum_{i=1}^{5} E_i - 2H) = -10 < 0$, so lemma 3.17 implies that $\mathrm{h}^1(\mathcal{O}_S(2H - \sum_{i=1}^{5} E_i)) = 0$ and thus also the desired vanishing. The rest of the claim follows by induction on k since $\mathcal{O}_H(k)$, for $k \geq 3$, is non-special for the general hyperplane section of S.□

We describe now the multiplicative structure of the dual of the H^1-module of $\mathcal{I}_S$. Let $V \subset \check{\mathbb{P}}^4$ denote the locus of hyperplanes, where the multiplication map $\mathrm{H}^1(\mathcal{I}_S(3)) \xrightarrow{H} \mathrm{H}^1(\mathcal{I}_S(4))$ has not maximal rank.

Lemma 3.19. $V \subset \check{\mathbb{P}}^4$ *is a quintic hypersurface; namely the variety of trisecant lines to an elliptic quintic scroll in* $\check{\mathbb{P}}^4$.

Proof. From the liaison exact sequences it follows that $\mathrm{h}^0(\mathcal{I}_S(4)) = 0$, thus Riemann-Roch and lemma 3.18 yield $\mathrm{h}^1(\mathcal{I}_S(3)) = \mathrm{h}^1(\mathcal{I}_S(4)) = 5$ and consequently V is a quintic hypersurface in $\check{\mathbb{P}}^4$. Now for the general plane π containing a ruling f of the elliptic scroll Q one has $\mathrm{h}^1(\mathcal{I}_{\pi\cap S}(4)) > 0$, and thus also $\mathrm{h}^1(\mathcal{I}_H(4)) > 0$ for the general hyperplane through f because $\mathrm{h}^2(\mathcal{I}_H(3)) = \mathrm{h}^1(\mathcal{O}_H(3)) = \mathrm{h}^1(\mathcal{O}_S(3)) = 0$. In conclusion, V is the dual variety of the elliptic scroll Q and therefore it is the variety of the trisecant lines to an elliptic quintic scroll in $\check{\mathbb{P}}^4$ (cf. [Seg]).□

The equation of the variety of trisecant lines to an elliptic quintic scroll was determined in [ADHPR]. Namely, in suitable coordinates $y_0 \ldots, y_4$ of $\check{\mathbb{P}}^4$, one has

$$V = \{y \mid \det\, (y_{3i+3j} z_{i-j})_{i,j\in\mathbb{Z}_5} = 0\},$$

for some parameter $z \in \mathbb{P}^4$, defined by $z_0 = 2$, $z_1 = a$, $z_2 = \frac{1}{a}$ and $z_i = z_{-i}$. (See also chapter 7 for more information on the Moore matrices $\{y_{3i+3j} z_{i-j}\}$).

Remark 3.20. *We have not been able to give examples, or give proof they do not exist, of rational surfaces as in the proposition* 3.6.

Proposition 3.21. *If S is a smooth surface of degree* 11, *with $\pi = 10$, $\chi = 2$ in $\mathbb{P}^4$, then S is a regular, minimal proper elliptic surface and its elliptic fibration is given by $|2K|$.*

Proof. Since $p_g \geq 1$ and $HK = 7 > 0$, S is either proper elliptic or of general type. Assume that S is a surface of general type and let S_0 denote the minimal model of S and

K_0 denote its canonical divisor. If S has at least two (-1) curves, then $HK_0 \leq 5$ and $p_a(K_0) = K_0^2 + 1 \geq 3$. Thus, necessarily $HK_0 = 5$, $p_a(K_0) = 3$, whereas, by (0.34), K_0 decomposes as the union of a plane quartic curve A and a line L which meet in one point. Now $K_0^2 = 2$ and $A^2 \leq 1$ by the index theorem, so $L^2 \geq -1$, which means that L is a (-1) line on S_0, a contradiction. Therefore S has only one (-1) line or conic E because otherwise $HK_0 \leq 4$, while $p_a(K_0) = K_0^2 + 1 = 2$, which is impossible by (0.34). If E is a (-1) conic, then $HK_0 = 5$, $p_a(K_0) = 2$, so K_0 spans only a hyperplane in $\mathbb{P}^4$, unless it splits as the union of a plane quartic A and a line L which don't meet. Now $A^2 + L^2 = 1$ and $A^2 \leq 1$ by Hodge index, so this case leads as above to a contradiction. Thus K_0 spans only a hyperplane and hence there is a residual curve $C \in |H - K_0|$. This curve has degree 6 and arithmetic genus $p_a(C) = 5$, so by (0.34) it is the union of a plane quintic Q and a line L which don't meet. But $C^2 = 2$ and $A^2 \leq 2$ by the index theorem, hence $L^2 \geq 0$, which is again absurd.
It follows that $K_0^2 = 1$ and S has only one exceptional line E. Moreover S is regular, otherwise $K_0^2 \geq 2p_g \geq 4$ by [Deb, Th.6.1] and S would have at least four (-1) curves. In particular we obtain that $p_g = 1$. For such a surface, $|2K_0|$ has no base points (see [Ca]) and $\Phi = \Phi_{|2K_0|} : S_0 \to \mathbb{P}^2$ is a morphism of degree 4. Let $\widetilde{\Phi} : S \to \mathbb{P}^2$ denote the composition of Φ with the blowing-down mapping $S \to S_0$. The restriction of $\widetilde{\Phi}$ to a hyperplane section H of S has degree 4 on the image because through four given points of S goes always at least one hyperplane section H. Therefore $\widetilde{\Phi}(H) \subset \mathbb{P}^2$ is a cubic curve and hence $|6K_0 - H| \neq \emptyset$. Moreover, since an irreducible cubic has at most one double point, there exists at most one (-2) curve F on S_0 of degree ≤ 2 in $\mathbb{P}^4$. On the other hand $K_0(6K_0 - H) = 0$, so a divisor $D \in |6K_0 - H|$ is a sum of E and F with certain multiplicities. But $E(6K_0 - H) = -1$ and $F(6K_0 - H) = -FH \geq -2$, so the multiplicities are 1 or 2, while $H(6K_0 - H) = 25$, which is again a contradiction.
For S an elliptic surface, the assertions of the proposition follow from the double point formula and Kodaira's formula [BPV] for the canonical class of an elliptic fibration.□

Remark 3.22. *We have not been able to give examples, or give proof they do not exist, of surfaces as in the above proposition.*

IV. Surfaces with $d = 11$, $\pi = 11$

The double point formula gives $K^2 = 6\chi - 17$, so Hodge index implies $\chi \leq 7$, or equivalently $\chi \leq 4$. For $\kappa(S) \geq 0$, since K is pseudo-effective, (0.16) yields then $\chi \in \{2, 3, 4\}$. In case $\kappa(S) = -\infty$, lemma 0.20 shows that the only possible value is $\chi = 1$.

A first example of a smooth rational surface with these invariants was constructed in [DES]. We recall here the argument and provide constructions for two further examples.

Proposition 3.23. *There exist smooth rational surfaces $S \subset \mathbb{P}^4$, with $d = 11$, $\pi = 11$ and embedded by one of the following linear systems*

$$a) \qquad H = 11l - 5E_0 - \sum_{i=1}^{6} 3E_i - \sum_{j=7}^{12} 2E_j - \sum_{k=13}^{19} E_k$$

$$b) \qquad H = 10l - 4E_0 - \sum_{i=1}^{3} 3E_i - \sum_{j=4}^{13} 2E_j - \sum_{k=14}^{19} E_k$$

$$c) \qquad H = 13l - 5E_0 - \sum_{i=1}^{7} 4E_i - \sum_{j=8}^{10} 2E_j - \sum_{k=11}^{19} E_k.$$

Proof. We construct the surfaces via the Eagon-Northcott method, so we need first to determine cohomology:

Lemma 3.24. *If $S \subset \mathbb{P}^4$ is a smooth surface with $d = 11$, $\pi = 11$, $\chi = 1$, then $\mathrm{h}^1(\mathcal{O}_S(1)) = 3$, $\mathrm{h}^1(\mathcal{O}_S(2)) = 1$ and $\mathrm{h}^1(\mathcal{O}_S(k)) = 0$, for all $k \geq 3$.*

Proof. Severi's theorem gives $\mathrm{h}^1(\mathcal{O}_S(1)) = 3$ and, since $\mathrm{h}^1(\mathcal{O}_H(2)) = \mathrm{h}^0(\mathcal{O}_H(K - H)) = 0$ for irreducible hyperplane sections, we deduce that $a = \mathrm{h}^1(\mathcal{O}_S(2)) \leq 3$. On the other side, Riemann-Roch and (1.6) yield $\mathrm{h}^1(\mathcal{O}_S(2)) \geq 1$. Consider now a hyperplane section H for which $\mathrm{h}^1(\mathcal{O}_H(2H)) \geq 1$. By lemma 0.37, we can find a decomposition $H = C_1 + C_2$, with $C_1 \geq 0$, $C_2 > 0$ and $C_1C_2 \leq (K - H)C_2$. We deduce

$$2 \deg C_2 \leq 2p_a(C_2) - 2 \qquad (*)$$

and the equivalent inequality

$$2 \deg C_1 \geq 2 + (K + H)C_1 + C_1C_2. \qquad (**)$$

By lemma 0.36, any plane curve on S has degree at most five, and the bounds in (0.34) show that the inequality $(*)$ is impossible, unless C_2 is a plane quintic, or $\deg C_2 \geq 8$. On the other side, since $(K + H)C_1 \geq 0$, it follows from $(**)$ that $\deg C_1 \geq 2$. In fact, in case $\deg C_1 = 2$, the 2-connectedness of the hyperplane sections [VdV] implies that $(K + H)C_1 = 0$, and thus, by (0.13), that C_1 is the union of two (-1) lines (which can also coincide), while $C_1^2 = HC_1 - C_1C_2 = 0$, and this is a contradiction. In case $\deg C_1 = 3$, we obtain again equality in $(*)$, and namely $\deg C_2 = 8$, $p_a(C_2) = 9$. Furthermore $C_1^2 + 1 = (K + H)C_1 \geq 0$ and thus $C_1^2 \geq -1$, whereas $KC_1 = 1 - C_1C_2 \leq -2$ by formula (0.26) and $C_1^2 \leq 0$ by Hodge index. It follows that $C_1^2 = 0$, so $\mathrm{h}^0(\mathcal{O}_S(C_1)) \geq \chi(\mathcal{O}_S(C_1)) \geq 2$ and thus C_2 would be a plane curve, which is absurd. Therefore, if $a \geq 2$, it follows that S would have a pencil of plane quintics. But this is impossible by

Lemma 3.25. *S has only finitely many plane curves.*

Proof. Since any pencil of plane curves on S is linear, the residual curve in a hyperplane section containing a general element of the family is again a plane curve. But then the hyperplane section is contained in a quadric and this contradicts Severi's theorem.□

Assume now that $\mathrm{h}^1(\mathcal{O}_S(3)) \geq 1$. Then $\mathrm{h}^1(\mathcal{O}_H(3)) \geq 1$ for at least a web of hyperplane sections H. But the general element in the web is smooth while $\mathcal{O}_H(3H)$ is non-special since $2\pi - 2 < 33$, and this is a contradiction.□

Corollary 3.26. *Let $\pi \subset \mathbb{P}^4$ be the base locus plane of the pencil of hyperplanes for which $\mathrm{h}^1(\mathcal{O}_H(2)) \neq 0$. Then $\pi \cap S$ contains as component a plane quintic curve. Residual to it, there is a pencil of smooth irreducible curves of degree 6, and arithmetic genus ≤ 1. Moreover, when the genus is one, the plane π cuts S along a plane quintic curve with an embedded point, and thus the surface has infinitely many 6-secant lines in this case.*

Proof. It follows from the proof of the above lemma that $\pi \cap S$ contains as component a plane quintic curve C. Let $|D| = |H - C|$ denote the residual pencil. Then the genus formula (0.2) gives $p_a(D) = D^2$, while Hodge index yields the bound $D^2 \leq 3$. In particular, the general member in the pencil $|H - C|$ is smooth. We show in the sequel that the cases $D^2 \in \{2, 3\}$ cannot occur.

Assume first that $D^2 = 3$. Then $CD = 3$, so the restriction of $|D|$ defines a linear system of degree 3 on C. Since C is not hyperelliptic, it follows that $|D|_{|C}$ is a g^1_3, and the genus formula tells us that C has, as its sole singularity, a node or an ordinary cusp, the g^1_3 being cut out by the pencil of lines through this double point. In particular $\mathrm{h}^0(\mathcal{O}_C(H - D)) \neq 0$, while the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{I}_H(1) \longrightarrow \mathcal{I}_D(1) \longrightarrow \mathcal{O}_C(H - D) \longrightarrow 0$$

gives $\mathrm{h}^0(\mathcal{O}_C(H - D)) = 0$ since, obviously, $\mathrm{h}^0(\mathcal{I}_D(1)) = 0$ and $\mathrm{h}^1(\mathcal{I}_H(1)) = 0$ by Severi's theorem, hence we've obtained a contradiction in this case.

If $D^2 = 2$, then $CD = 4$ so either $|D|$ cuts out a g^1_4 on C, or $|D|_{|C}$ has a base point P on C, where D is tangent to the plane π. In the last case, C is again singular and the free part of $|D|_{|C}$ is cut out by the pencil of lines through the double point. One argues now as in the previous case to obtain a contradiction.

Finally, when $D^2 = 1$ we obtain $CD = 5$ and then $|D|_{|C}$ must have a base point, otherwise, since a g^1_5 is cut out on C by the pencil of lines through a point outside the curve, the previous arguments would lead again to a contradiction. The claim of the corollary follows now easily.□

For construction purposes we assume $\mathrm{h}^0(\mathcal{I}_S(4)) = 0$, thus in other words that the cohomology diagram is minimal

$$
\begin{array}{c|c|c|c|c|c}
\uparrow i & & & & & \\
\hline
 & & & & & \\
\hline
 & & & & & \\
\hline
 & & 3 & 1 & & \\
\hline
 & & & & 2 & 1 \\
\hline
 & & & & & \\
\hline
 & & & & & \rightarrow p
\end{array}
\qquad \mathrm{h}^i(\mathcal{I}_S(p))
$$

Then Beilinson's spectral sequence yields for any such surface S a presentation

$$0 \longrightarrow 3\Omega^3(3) \xrightarrow{\varphi} \mathcal{G} \longrightarrow \mathcal{I}_S(4) \longrightarrow 0,$$

where $\mathcal{G}$ is the kernel of a morphism

$$0 \longrightarrow \mathcal{G} \longrightarrow \Omega^2(2) \oplus 2\Omega^1(1) \xrightarrow{\psi} \mathcal{O} \longrightarrow 0.$$

Identifying $\mathbb{P}^4 = \mathbb{P}(V)$, the morphism ψ is induced by a triple $(\psi_1, \psi_{21}, \psi_{22})$, with $\psi_1 \in \Lambda^2 V$ and $\psi_{21},\ \psi_{22} \in V$. It is easily seen that there are precisely four different choices which induce surjective ψ's:

1) $(\psi_1, \psi_{21}, \psi_{22}) \in \Lambda^2 V \times V \times V$ generic,

2) $\psi_1 = 0$ and $(\psi_{21}, \psi_{22}) \in V \times V$ generic,

3*a*) $\psi_1 \in \Lambda^2 V$ a decomposable tensor, $\psi_{21} \in V$ generic, and $\psi_{22} = 0$, or

3*b*) $\psi_1 \in \Lambda^2 V$ indecomposable, $\psi_{21} \in V$ generic, and $\psi_{22} = 0$.

Corresponding, we obtain four families of vector bundles $\mathcal{G}$ with minimal free resolutions of type

$$
\begin{array}{ccccccccc}
0 \leftarrow \mathcal{G}_1 \leftarrow 25\mathcal{O}(-1) & \nwarrow & 15\mathcal{O}(-2) & & 3\mathcal{O}(-3) & & & & \\
 & & \oplus & \longleftarrow & \oplus & \nwarrow & & & \\
 & & 2\mathcal{O}(-3) & & 3\mathcal{O}(-4) & & \mathcal{O}(-5) & \leftarrow 0, &
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & 25\mathcal{O}(-1) & & 16\mathcal{O}(-2) & & 4\mathcal{O}(-3) & & & \\
0 \leftarrow \mathcal{G}_2 \leftarrow & \oplus & \longleftarrow & \oplus & \longleftarrow & \oplus & \nwarrow & & \\
 & \mathcal{O}(-2) & & 3\mathcal{O}(-3) & & 3\mathcal{O}(-4) & & \mathcal{O}(-5) & \leftarrow 0,
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & 25\mathcal{O}(-1) & & 17\mathcal{O}(-2) & & 6\mathcal{O}(-3) & & \mathcal{O}(-4) & \\
0 \leftarrow \mathcal{G}_{3a} \leftarrow & \oplus & \longleftarrow & \oplus & \longleftarrow & \oplus & \longleftarrow & \oplus & \leftarrow 0, \\
 & 2\mathcal{O}(-2) & & 5\mathcal{O}(-3) & & 4\mathcal{O}(-4) & & \mathcal{O}(-5) &
\end{array}
$$

and respectively

$$0 \leftarrow \mathcal{G}_{3b} \leftarrow 25\mathcal{O}(-1) \nwarrow \begin{array}{c} 15\mathcal{O}(-2) \\ \oplus \\ 5\mathcal{O}(-3) \end{array} \longleftarrow \begin{array}{c} 6\mathcal{O}(-3) \\ \oplus \\ 4\mathcal{O}(-4) \end{array} \longleftarrow \begin{array}{c} \mathcal{O}(-4) \\ \oplus \\ \mathcal{O}(-5) \end{array} \leftarrow 0.$$

However, the choice $3b$) doesn't lead to any surface. Namely, in this case, $\mathcal{G}_{3b} = \Omega^1(1) \oplus \mathcal{H}$, where $\mathcal{H} = \ker(\Omega^2(2) \oplus \Omega^1(1) \stackrel{(\psi_1,\psi_{21})}{\longrightarrow} \mathcal{O})$, and there is no sheaf monomorphism $\varphi \in \mathrm{Hom}(3\Omega^3(3), \mathcal{G}_{3b})$ since in the minimal free resolution of $\mathcal{H}$

$$0 \leftarrow \mathcal{H} \leftarrow 15\mathcal{O}(-1) \nwarrow \begin{array}{c} 5\mathcal{O}(-2) \\ \oplus \\ 5\mathcal{O}(-3) \end{array} \longleftarrow \begin{array}{c} \mathcal{O}(-3) \\ \oplus \\ 4\mathcal{O}(-4) \end{array} \nwarrow \mathcal{O}(-5) \leftarrow 0,$$

the 5 linear syzygies of the 15 generators involve only 10 of these, and thus the component of φ going to $\mathcal{H}$ factorizes through a rank 4 sheaf. In all other three cases, by checking on a computer via [Mac], we find that a general $\varphi \in \mathrm{Hom}(3\Omega^3(3), \mathcal{G}_i)$, where $i \in \{1, 2, 3a\}$, defines a smooth surface $S_i \subset \mathbb{P}^4$ with the desired invariants.

In case 1) a mapping cone of the resolutions of $\mathcal{F} = 3\Omega^3(3)$ and $\mathcal{G}_1$ provides the minimal free resolution

$$0 \leftarrow \mathcal{I}_{S_1} \leftarrow 10\mathcal{O}(-5) \nwarrow \begin{array}{c} 12\mathcal{O}(-6) \\ \oplus \\ 2\mathcal{O}(-7) \end{array} \longleftarrow \begin{array}{c} 3\mathcal{O}(-7) \\ \oplus \\ 3\mathcal{O}(-8) \end{array} \nwarrow \mathcal{O}(-9) \leftarrow 0$$

and hence, in particular, we see that S_1 is cut out by quintic hypersurfaces and thus has no 6-secant lines. Let now Σ_1 denote the image of S_1 under the adjunction map, and Σ_2 denote the image of Σ_1 under the adjunction map defined by $|H_1 + K_1|$. Le Barz's formula gives $N_6 = 7$ so there are 7 exceptional lines on S_1, and we obtain from (0.13) the following invariants

$\Sigma_1 \subset \mathbb{P}^{10}$	$H_1^2 = 18$	$H_1K_1 = -2$	$K_1^2 = -4$	$\pi_1 = 9$
$\Sigma_2 \subset \mathbb{P}^8$	$H_2^2 = 10$	$H_2K_2 = -6$	$K_2^2 = -4 + b$	$\pi_2 = 3,$

where b is the number of (-1)-conics on S_1. Since $(H_2 + K_2)^2 = b - 6$ and the adjunction maps Σ_2 to $\mathbb{P}^2$, theorem 0.13 implies $b \in \{6, 7\}$. If $b = 6$, then Σ_2 is a conic bundle $\Sigma_2 \to \mathbb{P}^1$ with 6 singular fibres. Therefore $S_1 = \mathbb{F}_e(p_1, p_2, \ldots, p_{19})$ is a blown-up Hirzebruch surface and we can recover through adjunction its embedding

$$H = 6C_0 + (3e + 8)F - \sum_{i=1}^{6} 3E_i - \sum_{j=7}^{12} 2E_j - \sum_{k=13}^{19} E_k,$$

where C_0 denotes a section with self-intersection $-e \leq 0$ and F a fibre of the ruling. Now $HC_0 \geq 1$, so $e \in \{0, 1, 2\}$ and we may choose (via elementary transformations) $\mathbb{P}^2$ as minimal model, whence

$$H = 11l - 5E_0 - \sum_{i=1}^{6} 3E_i - \sum_{j=7}^{12} 2E_j - \sum_{k=13}^{19} E_k. \tag{1}$$

If $b = 7$, then Σ_2 is $\mathbb{P}^2$ blown-up in 6 points and we obtain

$$H = 10l - \sum_{i=1}^{6} 3E_i - \sum_{j=7}^{13} 2E_j - \sum_{k=14}^{20} E_k. \tag{2}$$

To exclude the linear system (2) we'll use the information provided by corollary 3.26. Namely, since S_1 has no 6-secant lines, it follows that the plane distinguished by the H^2-cohomology meets the surface along a plane quintic curve C, whose residual pencil $|D| = |H - C|$ is base point free and consists of rational curves of degree 6. Let now D_1 be the image of a general element in this pencil through the adjunction map on S_1. We have $H_1D_1 = (H + K)D = 4$, and the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_{S_1}(K) \longrightarrow \mathcal{O}_{S_1}(K + C) \longrightarrow \mathcal{O}_C(K + C) \longrightarrow 0$$

gives $\mathrm{h}^0(\mathcal{O}_{\Sigma_1}(H_1 - D_1)) = \mathrm{h}^0(\mathcal{O}_{S_1}(H + K - D)) = \mathrm{h}^0(\mathcal{O}_{S_1}(K + C)) = \mathrm{h}^0(\mathcal{O}_C(K + C)) = 6$. Therefore $D_1 \subset \mathbb{P}^{10}$ has degree 4 and spans a whole $\mathbb{P}^4$, thus it is a rational normal curve in the spanned linear subspace. The image of D_1 through the map defined by $|H_1 + K_1|$ is then a curve of degree $D_1(H_1 + K_1) = 2 - D_1^2 \leq 2$, thus $\Sigma_2 \subset \mathbb{P}^8$ is a conic bundle. It follows that in this case S_1 is a rational surface of type a), i.e., embedded by a linear system of type

$$H = 11l - 5E_0 - \sum_{i=1}^{6} 3E_i - \sum_{j=7}^{12} 2E_j - \sum_{k=13}^{19} E_k.$$

In case 2) the ideal sheaf of S_2 has syzygies

$$0 \leftarrow \mathcal{I}_{S_2} \leftarrow \begin{matrix} 10\mathcal{O}(-5) \\ \oplus \\ \mathcal{O}(-6) \end{matrix} \longleftarrow \begin{matrix} 13\mathcal{O}(-6) \\ \oplus \\ 3\mathcal{O}(-7) \end{matrix} \longleftarrow \begin{matrix} 4\mathcal{O}(-7) \\ \oplus \\ 3\mathcal{O}(-8) \end{matrix} \nwarrow \mathcal{O}(-9) \leftarrow 0\,.$$

This family was first constructed in [DES]. We recall in the sequel from [DES] the description of the embedding in $\mathbb{P}^4$. First observe that the H^1-module distinguishes a line L such that $I_{S_2}/(I_{S_2})_{\leq 5}$ has support on it. Thus L is the unique 6-secant, and by Le Barz's formula there are 6 exceptional lines on S_2. Denoting as above by Σ_1 and Σ_2 the first and the second adjoint surfaces of S_2 respectively, we obtain via (0.13) the following invariants

$\Sigma_1 \subset \mathbb{P}^{10}$	$H_1^2 = 18$	$H_1K_1 = -2$	$K_1^2 = -5$	$\pi_1 = 9$
$\Sigma_2 \subset \mathbb{P}^8$	$H_2^2 = 9$	$H_2K_2 = -7$	$K_2^2 = -5 + b$	$\pi_2 = 2$,

where b is the number of (-1)-conics on S_2. It follows that $(H_2 + K_2)^2 = 0$, thus $b = 10$, and the next adjunction morphism presents Σ_2 as a conic bundle with 3 singular fibres. Therefore $S_2 = \mathbb{F}_e(p_1, p_2, \ldots, p_{19})$ is a blown-up Hirzebruch surface and

$$H = 6C_0 + (3e + 7)F - \sum_{i=1}^{3} 3E_i - \sum_{j=4}^{13} 2E_j - \sum_{k=14}^{19} E_k.$$

$HC_0 \geq 1$, so $e \in \{0,1,2\}$ and we may choose $\mathbb{P}^2$ as minimal model. In particular

$$H = 10l - 4E_0 - \sum_{i=1}^{3} 3E_i - \sum_{j=4}^{13} 2E_j - \sum_{k=14}^{19} E_k$$

and S_2 is a surface of type b). We remark that in this case the pencil described in corollary 3.26 is the pullback of the conic fibration on Σ_2.

Finally, in case 3) we obtain a surface S_3 with syzygies

$$0 \leftarrow \mathcal{I}_{S_3} \leftarrow \begin{array}{c} 10\mathcal{O}(-5) \\ \oplus \\ 2\mathcal{O}(-6) \end{array} \longleftarrow \begin{array}{c} 14\mathcal{O}(-6) \\ \oplus \\ 5\mathcal{O}(-7) \end{array} \longleftarrow \begin{array}{c} 6\mathcal{O}(-7) \\ \oplus \\ 4\mathcal{O}(-8) \end{array} \longleftarrow \begin{array}{c} \mathcal{O}(-8) \\ \oplus \\ \mathcal{O}(-9) \end{array} \longleftarrow 0$$

The quintics in the ideal cut out the surface S_3 plus the plane π distinguished by the H^2-cohomology, namely $\pi = \mathrm{span}_k(e_0, e_1, e_2)$, if $V = \mathrm{span}_k(e_0, \ldots, e_4)$ and $\psi_1 = e_1 \wedge e_2$, $\psi_{12} = e_0$ in the above construction. π meets S_3 along a plane quintic curve C with an embedded point $P = \mathbb{P}(ke_0)$, so the surface has infinitely many 6-secant lines. The residual pencil $|D| = |H - C|$ has only one base point at P and consists of elliptic curves. One checks in an example that S_3 has 9 exceptional lines. The adjunction process produces surfaces with the following invariants

$\Sigma_1 \subset \mathbb{P}^{10}$	$H_1^2 = 18$	$H_1K_1 = -2$	$K_1^2 = -2$	$\pi_1 = 9$
$\Sigma_2 \subset \mathbb{P}^8$	$H_2^2 = 12$	$H_2K_2 = -4$	$K_2^2 = -2 + b$	$\pi_2 = 5,$

where b is the number of (-1)-conics on S_3. Now Hodge index gives $b \leq 3$, so the next adjoint surface

$\Sigma_3 \subset \mathbb{P}^4$	$H_3^2 = 2 + b$	$H_3K_3 = -6 + b$	$K_3^2 = -2 + b + c$	$\pi_3 = b - 1$

where c is the number of (-1)-twisted cubics on S_3, will be a surface of degree at most 5. Let again D_1 be the image on Σ_1 of a general element in the pencil $|D|$. Then $H_1D_1 = 5$ and as above D_1 spans a $\mathbb{P}^4$, thus $p_a(D_1) \leq 1$. If $p_a(D_1) = 0$, then $|D_1|$ has two base points, hence $D_1^2 = 2$, $K_1D_1 = -4$ and thus $|D_1|$ would map through the second adjunction on a pencil of lines, absurd. Therefore $|D_1|$ is a pencil of elliptic curves of degree 5 with only one base point. Similar arguments show in fact that $|D|$ maps via adjunction on a pencil of plane cubic curves on Σ_3. Since the rational cubic scroll and the Veronese surface do not possess such pencils, it follows that $b = 3$ and Σ_3 is a Castelnuovo surface (see e.g. [Ok2]), i.e., a surface in $\mathbb{P}^4$ embedded by

$$H_{\Sigma_3} = 4l - 2E_0 - \sum_{i=1}^{7} E_i.$$

Going back through the adjunction process we recuperate the initial embedding in $\mathbb{P}^4$

$$H = 13l - 5E_0 - \sum_{i=1}^{7} 4E_i - \sum_{j=8}^{10} 2E_j - \sum_{k=11}^{19} E_k,$$

thus S_3 is a surface of type $c)$ this time. We remark that in this case $|D| = |3l - \sum_{i=0}^{7} E_i|$.□

Remark 3.27. The above families of rational surfaces lie in different components of the Hilbert scheme. The Eagon-Northcott method used in the previous proposition allows to construct also schemes lying at the intersection of two such components. For example, let $\rho : \mathbb{P}^1 \times \mathbb{P}^4 \to \mathbb{P}^4$ be the projection onto the second factor and consider the bundle $\mathfrak{G}$ on $\mathbb{P}^1 \times \mathbb{P}^4$ defined as

$$\mathfrak{G} = \ker(\rho^*\Omega^2(2) \oplus 2\rho^*\Omega^1(1) \xrightarrow{\psi} \mathcal{O}),$$

where $\psi = (te_0 \wedge e_2, se_0, te_1)$, and s, t denote the coordinates in $\mathbb{P}^1$. Then $\mathfrak{G}_{(s:1)}$ is a bundle on $\mathbb{P}^4$ of type $\mathcal{G}_2$, for all $s \neq 0$, while $\mathfrak{G}_{(0:1)}$ is of type $\mathcal{G}_{3a}$. The degeneracy locus of a general morphism $\varphi \in \mathrm{Hom}(\rho^*\Omega^3(3), \mathfrak{G})$ defines an irreducible flat family of surfaces of degree 11, $\pi = 11$, $\chi = 1$ in $\mathbb{P}^4$, with general fibre a smooth rational surface of type $b)$, and with the fibre over (0:1) a scheme lying in the intersection of the components of the Hilbert scheme containing rational surfaces of type $b)$ and $c)$, respectively. Explicit computations in one example, via [Mac], show that a general such scheme is the union of the plane $\mathrm{span}_k(e_0, e_1, e_2)$ and a smooth rational surface of degree 10, sectional genus 8 meeting the plane along a quartic curve.□

Proposition 3.28. *Let S be a smooth surface in $\mathbb{P}^4$ with invariants $d = 11$, $\pi = 11$ and $\chi = 2$. Then either*
a) S is a blown-up $K3$ surface, embedded by

$$H = H_{min} - 5E_1 - \sum_{i=2}^{5} E_i,$$

or
b) S is a blown-up $K3$ surface, embedded by

$$H = H_{min} - 4E_1 - 2E_2 - \sum_{i=3}^{5} E_i,$$

or
c) S is a blown-up $K3$ surface, embedded by

$$H = H_{min} - 3E_1 - \sum_{i=2}^{3} 2E_i - \sum_{j=4}^{5} E_j,$$

or
d) S is a blown-up $K3$ surface, embedded by

$$H = H_{min} - \sum_{i=1}^{4} 2E_i - E_5.$$

Proof. We use adjunction to produce a number of possible candidates for the embedding $|H|$ of S in $\mathbb{P}^4$. Let S_1 denote the image of S under the adjunction map, and S_2 denote the image of S_1 under the adjunction map defined by $|H_1 + K_1|$. We obtain from (0.13) the following invariants

$S \subset \mathbb{P}^4$	$H^2 = 11$	$HK = 9$	$K^2 = -5$	$\pi = 11$
$S_1 \subset \mathbb{P}^{11}$	$H_1^2 = 24$	$H_1K_1 = 4$	$K_1^2 = -5 + a$	$\pi_1 = 15$
$S_2 \subset \mathbb{P}^{15}$	$H_2^2 = 27 + a$	$H_2K_2 = a - 1$	$K_2^2 = -5 + a + b$	$\pi_2 = a + 14,$

where a is the number of (-1)-lines on S and b is the number of (-1)-lines on S_1. Since $(H_1K_1)^2 < H_1^2$, lemma 0.18 shows that $\kappa(S) \in \{0, 1\}$, and thus S is either a $K3$ surface or a proper elliptic surface. In particular $a \leq 5$ and $a + b \leq 5$. On the other hand $H_2K_2 = a - 1 \geq 0$, so S has at least one exceptional line. Now, in case S has at most three (-1) lines, then $HK_{min} \leq HK - a - 2b - \sharp(\text{exceptional curves of higher degree}) \leq 2$, hence S is necessarily a $K3$ surface and H, easily reconstructed via the adjunction, is one in the following list of candidates:

$$1) \qquad H = H_{min} - \sum_{i=1}^{4} 2E_i - E_5$$

$$2) \qquad H = H_{min} - 3E_1 - \sum_{i=2}^{3} 2E_i - \sum_{j=4}^{5} E_j$$

$$3) \qquad H = H_{min} - \sum_{i=1}^{2} 3E_i - \sum_{j=3}^{5} E_j$$

$$4) \qquad H = H_{min} - 4E_1 - 2E_2 - \sum_{i=3}^{5} E_i.$$

When S has four exceptional lines, then $K_1^2 = -1$ and $H_2K_2 = 3$. Therefore, either S_2 is minimal and S is a regular, proper elliptic surface embedded by

$$5) \qquad H = H_{min} - 2E_1 - \sum_{i=2}^{5} E_i, \quad \text{(elliptic)}$$

or S is a blown-up $K3$ surface embedded by

$$6) \qquad H = H_{min} - 5E_1 - \sum_{i=2}^{5} E_i. \qquad (K3)$$

Finally, if S has five exceptional lines, then S_1 is minimal, whence S is a proper elliptic surface embedded by

$$7) \qquad H = H_{min} - \sum_{i=1}^{5} E_i. \quad \text{(elliptic)}$$

We go on to study these candidates, excluding all but 1), 2), 4) and 6). First a lemma

Lemma 3.29. *If $S \subset \mathbb{P}^4$ is a smooth surface with $d = 11$, $\pi = 11$ and $\kappa(S) \geq 0$, then $\mathrm{h}^1(\mathcal{O}_S(k)) = 0$, for all $k \geq 2$.*

Proof. We observe that $\mathrm{h}^2(\mathcal{O}_S(n)) = \mathrm{h}^0(\mathcal{O}_S(K - nH)) = 0$ for all $n \geq 1$ because $(K - nH)(K + H) = 6\chi - 8 - 20n < 0$. Riemann-Roch and Severi's theorem then give $\mathrm{h}^1(\mathcal{O}_S(1)) \leq 2$. On the other side $\mathrm{h}^1(\mathcal{O}_H(2)) = \mathrm{h}^0(\mathcal{O}_H(K - H)) = 0$ since $H(K-H) < 0$, thus $\mathrm{h}^1(\mathcal{O}_S(k)) \geq \mathrm{h}^1(\mathcal{O}_S(k+1))$ for all $k \geq 1$, and therefore to prove the lemma it is enough to check that $\mathrm{h}^1(\mathcal{O}_S(2)) = 0$. Assume that $\mathrm{h}^1(\mathcal{O}_S(2)) > 0$. Then the variety $V \subset \check{\mathbb{P}}^4$ parametrizing hyperplane sections for which $\mathrm{h}^1(\mathcal{O}_H(2)) > 0$ contains a plane, so there is a line L contained in a net of hyperplanes for which $\mathrm{h}^1(\mathcal{O}_H(2))$ doesn't vanish. The line L is contained in S and gives rise to a residual exact sequence

$$0 \longrightarrow \mathcal{O}_C(2H - L) \longrightarrow \mathcal{O}_H(2H) \longrightarrow \mathcal{O}_L(2H) \longrightarrow 0.$$

If the general section $C \in |H - L|$ is irreducible, then the cohomology of the above exact sequence implies that $2p_a(C) - 2 \geq (H - L)(2H - L)$, which combined with the genus formula says that $L^2 \geq -1$, whereas $L^2 = -1$ because of the assumption on the Kodaira dimension. We obtain $p_a(C) = 10$, and then the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_H(2H) \longrightarrow \mathcal{O}_C(2H) \longrightarrow 0$$

yields $\mathrm{h}^1(\mathcal{O}_C(2H)) > 0$, which is absurd since C is irreducible and $2HC > 2p_a(C) - 2 = 18$. The case when $|H - L|$ is composed with a pencil can be ruled out as in (3.4). Therefore $\mathrm{h}^1(\mathcal{O}_S(2)) = 0$ and the claim of the lemma follows.□

From the above lemma, combined with Riemann-Roch and theorems 1.6 and 1.7, we deduce $\mathrm{h}^1(\mathcal{I}_S(2)) = \mathrm{h}^1(\mathcal{O}_S(2)) = 0$, and thus $\mathrm{h}^0(\mathcal{I}_H(3)) = 0$ for all hyperplane sections H of S.

In case 3), there exist residual curves $D_i \in |H - E_i|$, for $i = 1, 2$, of degree 8 and arithmetic genus 8. Moreover, the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_S(H + E_i) \longrightarrow \mathcal{O}_S(2H) \longrightarrow \mathcal{O}_{D_i}(2H) \longrightarrow 0$$

yields $\mathrm{h}^1(\mathcal{O}_{D_i}(2)) = 0$ since $\mathrm{h}^2(\mathcal{O}_S(H + E_i)) = \mathrm{h}^0(\mathcal{O}_S(K - H - E_i)) \leq \mathrm{h}^0(\mathcal{O}_S(K - H)) = 0$. Thus Riemann-Roch implies that $\mathrm{h}^0(\mathcal{O}_{D_i}(2)) = 9$, and therefore either D_i lies on a smooth quadric surface Q_i, as a curve of type $(3, 5)$, or D_i splits as the union of two plane quartic curves which meet along a scheme of length 3. We sketch in the sequel only the former case, but similar arguments apply to rule out also the second one. Let H_1 and H_2 be the hyperplanes spanned by E_1 and E_2 respectively, and let $\pi = H_1 \cap H_2$ be their intersection plane. Since $D_1D_2 = 5$, it follows from Bezout's theorem that the two quadric surfaces share a curve on the plane π. Assume first that Q_1 and Q_2 meet both π along the same conic C. Then, since $E_iD_j = 3$, for $i \neq j$, we deduce that each twisted cubic E_i intersects the conic C in a scheme of length 3, and thus either π cuts S along eleven points on a conic, or C is contained in the surface S. In the former case the cohomology of the exact sequences

$$0 \longrightarrow \mathcal{I}_H(1) \longrightarrow \mathcal{I}_H(2) \longrightarrow \mathcal{I}_{\pi \cap S}(2) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{I}_S \longrightarrow \mathcal{I}_S(1) \longrightarrow \mathcal{I}_H(1) \longrightarrow 0$$

yields $\mathrm{h}^1(\mathcal{I}_S(1)) = \mathrm{h}^1(\mathcal{I}_H(1)) > 0$, which contradicts Severi's theorem. In the last case we may write $D_i = C + G_i$, $i = 1, 2$, with G_i curves of type $(2, 4)$ on the quadrics Q_i. In this setting we obtain $CG_i = 6$, $E_iC = 3$, $E_iG_j = 1$ and thus also $G_1G_2 = (H - E_1 - C)G_2 = -1$, which is absurd for two effective divisors without common components. At last, the case when the two quadrics Q_i, $i = 1, 2$, share only a line in the plane π, can be excluded in a similar way.

In case 5), the canonical divisor $K_{\min}$ of the minimal model of S would be a plane cubic curve in $\mathbb{P}^4$. Thus residual to it there is a pencil $|D| = |H - K_{\min}|$ of curves of degree 8 and arithmetic genus 8. As above one sees that $\mathrm{h}^0(\mathcal{I}_D(2)) \neq 0$. But then a hyperplane section of S by a hyperplane containing the plane of $K_{\min}$ will be contained in a cubic surface, thus contradicting the fact that $\mathrm{h}^0(\mathcal{I}_S(3)) = \mathrm{h}^1(\mathcal{I}_S(2)) = 0$.

In case 7), we obtain $HK_{\min} = HK_1 = 4$ this time, so the elliptic curve $K_{\min}$ spans only a $\mathbb{P}^3$. The residual curve D has degree 7 and arithmetic genus 7, thus splits by (0.34) as $D = A + B$, where A is a plane quintic curve and B is a conic meeting A along a scheme of length 2. This is a contradiction since, by (0.36), the degree of a plane curve on S is at most 4. We have proved therefore the claim of the proposition.□

(3.30.) Constructions. We'll show in the sequel that all the types of surfaces in the above proposition exist. For construction purposes we'll assume that S lies on no quartic hypersurface, i.e., that the cohomology table of $\mathcal{I}_S()$ looks like

i ↑

1				
	2			
			3	2

→ p

$\mathrm{h}^i(\mathcal{I}_S(p))$

Everything is determined by the module structure of the H^1-module of the ideal sheaf. Beilinson's theorem suggests in all cases to take

$$\mathcal{E} = \mathcal{O}(-1) \oplus 2\Omega^3(3) \qquad \text{and} \qquad \mathcal{F} = Syz_1(\mathrm{H}^1(\mathcal{I}_S(* + 4)))$$

and to construct the surface S as the degeneracy locus of a morphism $\psi \in \mathrm{Hom}(\mathcal{E}, \mathcal{F})$. Assuming that the k-dual of $\mathrm{H}^1(\mathcal{I}_S(*))$ is generated by elements in the first non-zero twist, a minimal free presentation of $\mathrm{H}^1(\mathcal{I}_S(*))^*$ is of type

$$0 \longleftarrow \mathrm{H}^1(\mathcal{I}_S(*))^* \longleftarrow 2R(5) \xleftarrow{\varphi} 7R(4) \oplus aR(3),$$

where $0 \leq a \leq 3$ depends on the number of linear syzygies of the linear part of φ. The morphism $\varphi = (\varphi_1, \varphi_2)$ is given by a $2 \times (7 + a)$ matrix with linear entries in φ_1 and quadratic in φ_2. For a generic choice, φ_1 has artinian cokernel and we may think of it as containing a 2×5 block whose cokernel is supported on 5 points. Assuming that the 5 points are the vertices $\{p_i\}$ of the standard simplex in k^5 and performing column operations on φ_1, we obtain that

$$\varphi_1 = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & 0 & 0 \\ a_0x_0 & a_1x_1 & a_2x_2 & a_3x_3 & a_4x_4 & l_1 & l_2 \end{pmatrix},$$

with l_i linear forms $l_i = \sum_{j=0}^{4} b_{ij}x_j$, $i = 1, 2$. We distinguish four cases:

a) l_1 and l_2 are two general linear forms, whereas $a = 0$, thus the presentation matrix is given by φ_1 alone in this case. For this choice, the bundle $\mathcal{F}_a$ has a free resolution of type

$$\begin{array}{ccccccc} & & & 10\mathcal{O}(-2) & & & \\ 0 \leftarrow \mathcal{F}_a \longleftarrow 20\mathcal{O}(-1) & \longleftarrow & & \oplus & \nwarrow & & \\ & & & 5\mathcal{O}(-3) & & 7\mathcal{O}(-4) \longleftarrow 2\mathcal{O}(-5) \leftarrow 0 & \end{array}$$

and one checks via [Mac] that the general $\psi \in \mathrm{Hom}(\mathcal{E}, \mathcal{F}_a)$ leads to a smooth surface $S_a \subset \mathbb{P}^4$ with desired invariants and whose ideal sheaf has a minimal free resolution

$$\begin{array}{ccccc} 0 \leftarrow \mathcal{I}_{S_a} \longleftarrow 9\mathcal{O}(-5) & \nwarrow & 8\mathcal{O}(-6) & & \\ & & \oplus & \nwarrow & \\ & & 5\mathcal{O}(-7) & & 7\mathcal{O}(-8) \longleftarrow 2\mathcal{O}(-9) \leftarrow 0\,. \end{array}$$

In particular, since the ideal $\mathcal{I}_{S_a}$ is generated by quintics, S_a has no 6-secants and hence $N_6(11, 11, 2) = 4$ is the number of exceptional lines on S_a. It follows from proposition 3.28 that S_a is non-minimal $K3$ surface embedded by

$$H = H_{min} - 5E_1 - \sum_{i=2}^{5} E_i.$$

This family of $K3$ surfaces was first constructed in [DES].

b) l_1 and l_2 have a common zero at exactly one of the points p_i. Then φ_1 has 6 linear syzygies, thus $a = 1$ and we take as φ_2 a general column matrix of quadrics. We get this time a bundle $\mathcal{F}_b$ with resolution of type

$$\begin{array}{cccccccc} & 20\mathcal{O}(-1) & & 11\mathcal{O}(-2) & & \mathcal{O}(-3) & & \\ 0 \leftarrow \mathcal{F}_b \longleftarrow & \oplus & \longleftarrow & \oplus & \longleftarrow & \oplus & \nwarrow & \\ & \mathcal{O}(-2) & & 6\mathcal{O}(-3) & & 7\mathcal{O}(-4) & & 2\mathcal{O}(-5) \leftarrow 0 \end{array}$$

and the general morphism $\psi \in \mathrm{Hom}(\mathcal{E}, \mathcal{F}_b)$ gives a smooth surface $S_b \subset \mathbb{P}^4$ with resolution of type

$$\begin{array}{cccccccc} & 9\mathcal{O}(-5) & & 9\mathcal{O}(-6) & & \mathcal{O}(-7) & & \\ 0 \leftarrow \mathcal{I}_{S_b} \longleftarrow & \oplus & \longleftarrow & \oplus & \longleftarrow & \oplus & \nwarrow & \\ & \mathcal{O}(-6) & & 6\mathcal{O}(-7) & & 7\mathcal{O}(-8) & & 2\mathcal{O}(-9) \leftarrow 0\,. \end{array}$$

This time there is precisely one sextic generator of $\mathcal{I}_{S_b}$, and the 6 linear forms in the first step of the resolution define a line L in the hyperplane dual to the point p_i, which is the support of $\mathcal{I}_{S_b}/(\mathcal{I}_{S_b})_{\leq 5}$ and thus the unique 6-secant of S_b. Consequently there are only $N_6 - 1 = 3$ exceptional lines, and hence S_b is a non-minimal $K3$ surface embedded by

$$H = H_{min} - 4E_1 - 2E_2 - \sum_{i=3}^{5} E_i.$$

c) l_1 and l_2 have a common zero at two of the points p_i. In this case φ_1 has 7 linear syzygies and we take φ_2 to be a general 2×2 matrix of quadrics ($a = 2$). The bundle $\mathcal{F}_c$ has a minimal free resolution

$$0 \leftarrow \mathcal{F} \longleftarrow \begin{array}{c} 20\mathcal{O}(-1) \\ \oplus \\ 2\mathcal{O}(-2) \end{array} \longleftarrow \begin{array}{c} 12\mathcal{O}(-2) \\ \oplus \\ 7\mathcal{O}(-3) \end{array} \longleftarrow \begin{array}{c} 2\mathcal{O}(-3) \\ \oplus \\ 7\mathcal{O}(-4) \end{array} \begin{array}{c} \\ \nwarrow \\ \end{array} \begin{array}{c} \\ \\ 2\mathcal{O}(-5) \leftarrow 0 \end{array}$$

and the general morphism $\psi \in \mathrm{Hom}(\mathcal{E}, \mathcal{F}_c)$ gives rise to smooth surface $S_c \subset \mathbb{P}^4$ with syzygies

$$0 \leftarrow \mathcal{I}_{S_c} \longleftarrow \begin{array}{c} 9\mathcal{O}(-5) \\ \oplus \\ 2\mathcal{O}(-6) \end{array} \longleftarrow \begin{array}{c} 10\mathcal{O}(-6) \\ \oplus \\ 7\mathcal{O}(-7) \end{array} \longleftarrow \begin{array}{c} 2\mathcal{O}(-7) \\ \oplus \\ 7\mathcal{O}(-8) \end{array} \begin{array}{c} \\ \nwarrow \\ \end{array} \begin{array}{c} \\ \\ 2\mathcal{O}(-9) \leftarrow 0\,. \end{array}$$

In this case the module $\mathcal{I}_{S_c}/(\mathcal{I}_{S_c})_{\leq 5}$ has support on two skew lines, L_1 and L_2, which are the 6-secants of S_c. It follows that there are only $N_6 - 2 = 2$ exceptional lines, and hence S_c is a non-minimal $K3$ surface embedded by

$$H = H_{min} - 3E_1 - \sum_{i=2}^{3} 2E_i - \sum_{j=4}^{5} E_j.$$

d) Finally l_1 and l_2 have a common zero at three of the points p_i, say p_0, p_1 and p_2. Now φ_1 has 8 linear syzygies and we take φ_2 to be given by a general 2×3 matrix of quadric forms. This leads to a vector bundle with resolution

$$0 \leftarrow \mathcal{F}_d \longleftarrow \begin{array}{c} 20\mathcal{O}(-1) \\ \oplus \\ 3\mathcal{O}(-2) \end{array} \longleftarrow \begin{array}{c} 13\mathcal{O}(-2) \\ \oplus \\ 8\mathcal{O}(-3) \end{array} \longleftarrow \begin{array}{c} 3\mathcal{O}(-3) \\ \oplus \\ 7\mathcal{O}(-4) \end{array} \begin{array}{c} \\ \nwarrow \\ \end{array} \begin{array}{c} \\ \\ 2\mathcal{O}(-5) \leftarrow 0 \end{array}$$

and the general morphism $\psi \in \mathrm{Hom}(\mathcal{E}, \mathcal{F}_d)$ degenerates along a smooth surface $S_d \subset \mathbb{P}^4$ with resolution

$$0 \leftarrow \mathcal{I}_{S_d} \longleftarrow \begin{array}{c} 9\mathcal{O}(-5) \\ \oplus \\ 3\mathcal{O}(-6) \end{array} \longleftarrow \begin{array}{c} 11\mathcal{O}(-6) \\ \oplus \\ 8\mathcal{O}(-7) \end{array} \longleftarrow \begin{array}{c} 3\mathcal{O}(-7) \\ \oplus \\ 7\mathcal{O}(-8) \end{array} \begin{array}{c} \\ \nwarrow \\ \end{array} \begin{array}{c} \\ \\ 2\mathcal{O}(-9) \leftarrow 0\,. \end{array}$$

The choice we've made distinguishes a plane, namely $\Pi = \mathrm{span}_k(p_0, p_1, p_2)$, which meets the surface S_d along a smooth plane quartic curve C and the 3 points p_0, p_1 and p_2 outside it. It follows that the 3 lines $L_{ij} = \mathrm{span}_k(p_i, p_j)$, for $0 \leq i < j \leq 2$, are 6-secant lines to the surface S_d, and they are the only ones since Π is the support of $\mathcal{I}_{S_d}/(\mathcal{I}_{S_d})_{\leq 5}$. Consequently there is only one exceptional line, and thus S_d is a non-minimal $K3$ surface embedded by

$$H = H_{min} - 2\sum_{i=1}^{4} E_i - E_5. \qquad \square$$

Proposition 3.31. *A smooth surface $S \subset \mathbb{P}^4$ with invariants $d = 11$, $\pi = 11$, $\chi = 3$ is either a minimal, general type surface on the Noether's line, or a regular, non-minimal surface of general type with only one exceptional line.*

Proof. The double point formula gives $K^2 = 1$, so S is of general type. S is a regular surface, hence $p_g = 2$, otherwise [Deb. Th.6.1] yields $K^2_{\min} \geq 6$, thus $HK_{\min} \leq 4$ while $p_a(K_{\min}) \geq 7$, which is absurd by (0.34). Assume now that S is not minimal and let $E_1, \dots, E_k$ denote the exceptional curves on S. If $k \geq 2$, then $HK_{\min} \leq 7$ and $p_a(K_{\min}) = K^2_{\min} + 1 \geq 4$ thus a curve $D \in |K_{\min} - E_1|$ would have degree at most 6 and arithmetic genus at least 4. The only possibility is that $k = 2$, $HE_1 = HE_2 = 1$, whereas D spans only a $\mathbb{P}^3$, $HD = 6$ and $p_a(D) = 4$. But then a curve in $|H - D|$ would have degree 5 and arithmetic genus 4, which is absurd. Therefore there exists only one exceptional curve E on S. Assume that $HE \geq 2$. Then a curve $D \in |K_{\min} - E|$ has degree ≤ 5 and arithmetic genus 3, thus it spans only a $\mathbb{P}^3$ and equality in fact holds. By (0.34) the residual curve $H - D$ splits as $H - D = A + B$, where A is a plane quintic curve and B is a line disjoint of A. It follows that $A^2 + B^2 = 2$, thus also $A^2 \geq 3$ since $B^2 \leq -1$ and this contradicts Hodge index. In conclusion, if S is not minimal, then it has only one exceptional line.$\square$

Proposition 3.32. *There exist smooth regular non-minimal surfaces of general type $S \subset \mathbb{P}^4$ with $d = 11$, $\pi = 11$, $\chi = 3$ and with only one exceptional line.*

Proof. For construction purposes we assume that S doesn't lie on a quartic hypersurface, and thus, by lemma 3.29 above, that the cohomology table of $\mathcal{I}_S(*)$ is minimal

i ↑

2				
	1			
		1	4	3

→ p

$\mathrm{h}^i(\mathcal{I}_S(p))$

Beilinson's theorem suggests to take

$$\mathcal{E} = 2\mathcal{O}(-1) \oplus \Omega^3(3) \qquad \text{and} \qquad \mathcal{F} = Syz_1(\mathrm{H}^1(\mathcal{I}_S(*+4)))$$

and to define S as the degeneracy locus of a morphism $\varphi \in \mathrm{Hom}(\mathcal{E}, \mathcal{F})$. Thus we have again to determine the module structure of the H^1-module of the ideal sheaf. We assume that $M = \mathrm{H}^1(\mathcal{I}_S(*))$ is monogenous, i.e., that M is the tensor product of a monogenous module M' with Hilbert function $(1, 4, 3)$ in 4 variables, say over $R' = k[x_0, \ldots, x_3]$, with the Koszul complex of one hyperplane, say x_4. However, it is easily seen that for the generic choice of such a module no morphism in $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$ can be injective. The trick is to choose M' special, and for that we start with four general lines L_1, L_2, L_3 and L_4 in $\mathbb{P}^3 = \mathrm{Proj}(k[x_0, \ldots, x_3])$ and define M'^* as the cokernel of

$$0 \longleftarrow M'^* \longleftarrow 3R'(4) \overset{(\theta_1,\theta_2)}{\longleftarrow} 8R'(3) \oplus R'(2),$$

where θ_1 is the product $\gamma\alpha$ of a random matrix $\gamma \in M_{3,4}(k)$ with the direct sum α of the four Koszul complexes built on the lines L_i, $i = \overline{1,4}$ in $\mathbb{P}^3$, while θ_2 is a column matrix of general quadrics. Tensoring M' with the Koszul complex of x_4 and sheafifing syzygies we obtain a bundle $\mathcal{F}$ with minimal resolution:

$$0 \leftarrow \mathcal{F} \longleftarrow \begin{matrix} 15\mathcal{O}(-1) \\ \oplus \\ 4\mathcal{O}(-2) \end{matrix} \longleftarrow \begin{matrix} 9\mathcal{O}(-2) \\ \oplus \\ 12\mathcal{O}(-3) \end{matrix} \longleftarrow \begin{matrix} \mathcal{O}(-3) \\ \oplus \\ 11\mathcal{O}(-4) \end{matrix} \nwarrow 3\mathcal{O}(-5) \leftarrow 0$$

and this time a general morphism $\varphi \in \mathrm{Hom}(\mathcal{E}, \mathcal{F})$ provides a smooth surface $S \subset \mathbb{P}^4$, with desired invariants and syzygies

$$0 \leftarrow \mathcal{I}_S \longleftarrow \begin{matrix} 8\mathcal{O}(-5) \\ \oplus \\ 4\mathcal{O}(-6) \end{matrix} \longleftarrow \begin{matrix} 8\mathcal{O}(-6) \\ \oplus \\ 12\mathcal{O}(-7) \end{matrix} \longleftarrow \begin{matrix} \mathcal{O}(-7) \\ \oplus \\ 11\mathcal{O}(-8) \end{matrix} \nwarrow 3\mathcal{O}(-9) \leftarrow 0$$

We check now what kind of surface we did obtain. By construction, it follows from the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{I}_S(2) \longrightarrow \mathcal{I}_S(3) \longrightarrow \mathcal{I}_H(3) \longrightarrow 0$$

that the hyperplane section H_0, cut out on S by the hyperplane spanned by the four lines, is the unique one such that $\mathrm{h}^0(\mathcal{I}_{H_0}(3)) = 1$. One checks that for a general choice of the construction data, the unique cubic containing H_0 is a smooth Del Pezzo surface $X \subset \mathbb{P}^3$. Now cohomology in a another twist of the above exact sequence yields $\mathrm{h}^0(\mathcal{I}_{H_0}(5)) = 11$, thus we may link H_0 in the complete intersection of X with a general quintic to a curve G of degree 4, arithmetic genus -3. On another side, by construction, $I_S/I_{S_{\leq 5}}$ is supported on the four lines, and so we might guess that they are the 6-secants of S. Indeed, G is the union of the four lines we've started with, and since we can always choose the basis of $\mathrm{Pic}(X)$ such that $X \subset \mathbb{P}^3$ is embedded by $H_X \sim 3l - \sum_{i=1}^{6} E_i$, whereas the lines L_i are of class E_i, $i = \overline{1,4}$ (see [Ha] or [GP2]), we deduce that $H_0 \sim 15l - \sum_{i=1}^{4} 6E_i - 5E_5 - 5E_6$. Intersection theory on X shows now that each line L_i is a 6-secant to H_0, and thus also to S, and these are the only 6-secant lines to S. Since Le Barz's formula gives $N_6(11, 11, 3) = 5$ we deduce that there is one exceptional line on the surface S. This proves the claim of the proposition.□

V. Surfaces with $d = 11$, $\pi = 12$

Adjunction and the double point formula give $HK = 11$ and $K^2 = 6\chi - 22$, thus Hodge index yields $\chi \le 5$. Now lemma 0.20 implies that $\chi \ge 0$, and equality holds only when S is a (blown-up) ruled surface over an elliptic curve. Moreover, when $\kappa(S) \ge 0$, the pseudo-effectiveness (0.16) of K gives $\chi \ge 2$.

Lemma 3.33. *There are no smooth surfaces $S \subset \mathbb{P}^4$ with $d = 11$, $\pi = 12$ which are birationally ruled over an elliptic curve.*

Proof. We consider, once again, the adjunction mapping whose image

$$S_1 = \varphi_{H+K}(S) \subset \mathbb{P}^{10} \qquad \text{has invariants} \qquad d_1 = 11,\ H_1K_1 = -11,\ \pi_1 = 1,$$

whence $K_1^2 = 0$, S_1 is geometrically ruled and φ_{H+K} is the contraction of 22 exceptional lines on S. Therefore, the formula for the canonical class of a ruled surface shows that the fibers of the ruling of S are embedded as (degenerated) twisted cubics in $\mathbb{P}^4$. Let F be such a ruling. The residual curve $D = H - F$ has $\deg D = 8$ and $p_a(D) = 10$, whence, according to lemma 0.34, it contains a plane sextic curve as component. But this contradicts the bound on the degree of a plane curve in (0.36).$\square$

Proposition 3.34. *There are no smooth rational surfaces $S \subset \mathbb{P}^4$, with $d = 11$ and $\pi = 12$.*

Proof. Assume such surfaces exist. Severi's theorem gives $\mathrm{h}^1(\mathcal{O}_S(1)) = 4$. On the other hand, if $\mathrm{h}^1(\mathcal{O}_H(2)) = \mathrm{h}^0(\mathcal{O}_H(K-H)) \ne 0$ for the general hyperplane section H of S, then, since $H(K-H) = 0$, it would follow from Weil's lemma (see [W] or [So]) that $K - H$ is trivial, which is absurd. Therefore $\mathrm{h}^1(\mathcal{O}_H(2)) = 0$ for the general H, and thus the long cohomology sequence of

$$0 \longrightarrow \mathcal{O}_S(1) \longrightarrow \mathcal{O}_S(2) \longrightarrow \mathcal{O}_H(2) \longrightarrow 0$$

gives $\mathrm{h}^1(\mathcal{O}_S(2)) \le 4$. Now $\mathrm{h}^0(\mathcal{I}_S(3)) = 0$ by (1.7) and $\mathrm{h}^2(\mathcal{O}_S(3)) = \mathrm{h}^0(\mathcal{O}_S(K-2H)) = 0$ because $p_g = 0$, hence $\mathrm{h}^1(\mathcal{O}_S(3)) \ge 1$ since by Riemann-Roch $\chi(\mathcal{I}_S(3)) = 1$. It follows that the variety $V \subset \check{\mathbb{P}}^4$ parametrizing hyperplane sections for which $\mathrm{h}^1(\mathcal{O}_H(3)) \ne 0$ is non-empty. Let H be a hyperplane section of S corresponding to a point of V. Then lemma 0.37 yields a decomposition $H = C_1 + C_2$, $C_1 \ge 0$, $C_2 > 0$ such that

$$C_1C_2 \le (K - 2H)C_2 \tag{$*$}$$

whence

$$3 \deg C_2 \le 2p_a(C_2) - 2. \tag{$**$}$$

This is readily seen to be impossible for $\deg C_2 \le 5$ by using the bounds for the arithmetic genus in (0.34). Now lemma 0.36 shows that plane curves on S have degree at most 5. Combined with the bounds in (0.34) this implies $\deg C_2 \ge 9$, whence $\deg C_1 \le 2$. But the inequality $(*)$ is equivalent to

$$3 \deg C_1 \ge (K + H)C_1 + C_1C_2 + 11.$$

Therefore, since $(K+H)C_1 \geq 0$ by (0.16) and $C_1C_2 \geq 1$ by the 1-connectedness of H, we obtain $\deg C_1 \geq 4$, which is a contradiction. $\square$

We discuss in the sequel the possible invariants for a surface S with $\kappa(S) \geq 0$.

Proposition 3.35. *Let $S \subset \mathbb{P}^4$ be a smooth surface with $d = 11$, $\pi = 12$ and $\chi = 2$. Then S is a non-minimal $K3$ surface embedded by*

$$H = H_{\min} - 2E_1 - \sum_{i=2}^{10} E_i.$$

Proof. Let S_1 be the image of S through the adjunction map. We compute the following invariants for $S_1 \subset \mathbb{P}^{12}$: $d_1 = 23$, $\pi_1 = 13$, $H_1K_1 = 1$. Now $p_g \geq 1$, so $|K_1|$ consists of a unique exceptional line and S is a blown-up $K3$ surface. In particular $K_1^2 = -1$ and the adjunction morphism φ_{H+K} is blowing down 9 exceptional lines on S.$\square$

An example of a smooth surface with the above invariants has been first constructed via liaison by K. Ranestad (private communication). We'll recall here his approach and prove that the general surface can be obtained by this construction. First a remark:

Lemma 3.36. *Let $S \subset \mathbb{P}^4$ be a surface with $d = 11$, $\pi = 12$ and $\kappa(S) \geq 0$. Then*
a) $\mathrm{h}^1(\mathcal{O}_S(1)) > \mathrm{h}^1(\mathcal{O}_S(2))$ *or* $\mathrm{h}^1(\mathcal{O}_S(2)) = 0$
b) $\mathrm{h}^1(\mathcal{O}_S(k)) = 0$ *for* $k \geq 3$.

Proof. First observe that $\mathrm{h}^2(\mathcal{O}_S(n)) = \mathrm{h}^0(\mathcal{O}_S(K - nH)) = 0$ for all $n \geq 1$ because $(K - nH)(K+H) = 6\chi - 11(2n+1) < 0$. Thus Severi's theorem and Riemann-Roch give for the speciality $\mathrm{h}^1(\mathcal{O}_S(1)) = 5 - \chi \leq 3$. The argument used in the proof of proposition 3.34 gives here $\mathrm{h}^1(\mathcal{O}_H(2)) = 0$ for a general hyperplane section H, hence $\mathrm{h}^1(\mathcal{O}_S(2)) \leq \mathrm{h}^1(\mathcal{O}_S(1)) = 5 - \chi$. Assume first that $\mathrm{h}^1(\mathcal{O}_S(2)) = \mathrm{h}^1(\mathcal{O}_S(1))$. Then the variety $V \subset \check{\mathbb{P}}^4$ parametrizing hyperplane sections for which $\mathrm{h}^1(\mathcal{O}_H(2)) \neq 0$ is a determinantal hypersurface of degree ≤ 3. It is not contained in the dual variety of S since otherwise S would be degenerated when $\deg V = 2$ or would have too many plane curves when $\deg V = 3$. Therefore one can find a Lefschetz pencil δ (see [AF], [Z]) such that for H a general member in it $\mathrm{h}^1(\mathcal{O}_H(2)) \neq 0$ and hence $\mathcal{O}_H(K - H)$ is trivial. But then Weil's lemma applies again to show that $K - H$ is trivial, which is absurd. It follows that $\mathrm{h}^1(\mathcal{O}_S(2)) \leq 4 - \chi \leq 2$ or $\mathrm{h}^1(\mathcal{O}_S(2)) = 0$.
We show next that $\mathrm{h}^1(\mathcal{O}_S(k)) = 0$ for $k \geq 3$. First of all $\mathrm{h}^1(\mathcal{O}_H(k)) = 0$ when $k \geq 3$ and H is a smooth hyperplane section, thus the claim follows by induction for $\chi = 4, 5$. Assume now $\chi \leq 3$ and $\mathrm{h}^1(\mathcal{O}_S(3)) \neq 0$. Then the variety $W \subset \check{\mathbb{P}}^4$ parametrizing hyperplane sections for which $\mathrm{h}^1(\mathcal{O}_H(3)) \neq 0$ contains a plane, so there is a line $L \subset S$ which is the base locus of a net of hyperplanes for which $\mathrm{h}^1(\mathcal{O}_S(3)) \neq 0$. If the general divisor $C \in | H - L |$ is irreducible, then it follows from the exact sequence

$$0 \longrightarrow \mathcal{O}_C(3H - L) \longrightarrow \mathcal{O}_H(3H) \longrightarrow \mathcal{O}_L(3H) \longrightarrow 0$$

that $h^1(\mathcal{O}_C(3H-L)) \neq 0$, and thus $2p_a(C)-2 \geq (H-L)(3H-L)$. Combined with formula (0.2), which reads

$$22 = 2p_a(C) - 2 - 2L^2,$$

this gives $L^2 \geq 7$, which is impossible. If the general divisor $C \in\mid H-L\mid$ is not integral, then $\mid H-L \mid$ is composed with a basepoint free pencil of plane curves. We have $\mathcal{O}_C(3H-L) = \mathcal{O}_C(2H)$ and thus $h^1(\mathcal{O}_C(2H)) > 0$. It follows that $\mid H-L \mid$ is a pencil of plane quintic curves. This is a contradiction since

$$2p_a(C) - 2 = CK = HK - LK \leq 11 + 2 + L^2 = 4.$$

$\square$

Lemma 3.37. *Let $S \subset \mathbb{P}^4$ be a smooth surface as in proposition* 3.35*. Then* $h^1(\mathcal{I}_S(k)) = 0$ *for all $k \geq 3$.*

Proof. Riemann-Roch gives $\chi(\mathcal{I}_S(3)) = 0$, whence $h^1(\mathcal{I}_S(3)) = 0$ because $h^0(\mathcal{I}_S(3)) = 0$ by (1.7) and $h^1(\mathcal{O}_S(3)) = h^2(\mathcal{O}_S(3)) = 0$ by lemma 3.36. It follows then from the cohomology of the exact sequences

$$0 \longrightarrow \mathcal{I}_S(m-1) \longrightarrow \mathcal{I}_S(m) \longrightarrow \mathcal{I}_H(m) \longrightarrow 0, \qquad m = \overline{3,4}$$

that $h^1(\mathcal{I}_H(4)) = h^1(\mathcal{I}_S(4))$ and $h^1(\mathcal{I}_H(3)) = 2$ for all hyperplane sections H. Assume now that $h^1(\mathcal{I}_S(4)) > 0$. Then each hyperplane contains at least one plane π for which $h^1(\mathcal{I}_{\pi\cap S}(4)) > 0$. For the general hyperplane this plane section is a finite scheme of length 11. If it contains a subscheme of length 10 which is contained in a conic, then $h^1(\mathcal{I}_{\pi\cap S}(3)) \geq 3$ and thus also $h^2(\mathcal{I}_H(2)) = h^1(\mathcal{O}_H(2)) = h^0(\mathcal{O}_H(K-H)) \geq 1$, which in turn implies, as before, that $K-H$ is trivial, whence a contradiction. It follows from lemma 0.41 that $\pi \cap S$ contains a subscheme of length 6 which is contained in a line. But this means that the general hyperplane section to S contains 6-secant lines, which is absurd. Therefore $h^1(\mathcal{I}_S(4)) = 0$ and the lemma follows. $\square$

Corollary 3.38. *A smooth $K3$ surface $S \subset \mathbb{P}^4$ with $d = 11$, $\pi = 12$ is the degeneracy locus of a morphism φ*

$$0 \longrightarrow \mathcal{O}(-1) \oplus 3\Omega^3(3) \xrightarrow{\varphi} 2\Omega^2(2) \oplus 2\mathcal{O} \longrightarrow \mathcal{I}_S(4) \longrightarrow 0,$$

and thus its ideal sheaf has a minimal free resolution of type:

$$0 \leftarrow \mathcal{I}_S \longleftarrow \begin{matrix} 2\mathcal{O}(-4) \\ \oplus \\ 4\mathcal{O}(-5) \end{matrix} \nwarrow 7\mathcal{O}(-6) \longleftarrow 2\mathcal{O}(-7) \leftarrow 0.$$

Proof. The above lemmas give the cohomology table

i ↑

1				
	3	2		
				2

→ p

$\mathrm{h}^i(\mathcal{I}_S(p))$

hence the corollary follows from Beilinson's spectral sequence. □

Proposition 3.39. *The general smooth $K3$ surface $S \subset \mathbb{P}^4$ with $d = 11$, $\pi = 12$ is linked $(4,4)$ to a reducible surface $X_5 = P \cup P_1 \cup P_2 \cup P_3 \cup P_4$, where P is a plane and P_i are planes cutting P along lines l_i, $i = \overline{1,4}$, such that no three lines have common intersection points.*

Proof. Let $S \subset \mathbb{P}^4$ be the degeneracy locus of a general morphism

$$\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} \in \mathrm{Hom}(\mathcal{O}(-1) \oplus 3\Omega^3(3), 2\Omega^2(2) \oplus 2\mathcal{O}).$$

It follows from Severi's theorem and lemma 3.36 that the variety $W \subset \check{\mathbb{P}}^4$ parametrizing hyperplane sections for which $\mathrm{h}^1(\mathcal{O}_H(2)) \neq 0$ is a smooth rational cubic scroll in $\check{\mathbb{P}}^4$, namely the degeneration locus of the 2×3 matrix with linear entries which defines $\varphi_{22} \in \mathrm{Hom}(3\Omega^3(3), 2\Omega^2(2))$. The rulings of W, respectively its directrix, correspond to pencils of hyperplanes for which $\mathrm{h}^1(\mathcal{O}_H(2)) \neq 0$; the pencil parametrized by the directrix l being distinguished by the fact that

$$\dim\Big(\bigcap_{H \in \mathrm{H}^0(\mathcal{O}_l(1))^*} \ker(\mathrm{H}^1(\mathcal{O}_S(1)) \xrightarrow{\cdot H} \mathrm{H}^1(\mathcal{O}_S(2)))\Big) = 2,$$

where the above intersection is considered as a vector subspace of $\mathrm{H}^1(\mathcal{O}_S(1))$. In the case of the pencils parametrized by rulings of W this dimension is only one. Recall now from (3.35) that S is embedded by a linear system

$$H = H_{\min} - 2E_1 - \sum_{i=2}^{10} E_i.$$

Consider the residual pencil $|D| = |H - E_1|$. First of all, since $DE_1 = 3$, $|D|$ has no fixed component since this would lie in the plane of the conic E_1. Therefore the general

D is irreducible, of degree 9 and arithmetic genus 10. Moreover, it is in fact a complete intersection of two cubic surfaces since, by Riemann-Roch, $\mathrm{h}^0(\mathcal{I}_{D,H}(3)) \geq 2$ and since if D would lie on a quadric, then it would be a curve of type $(3,6)$ on a smooth quadric surface and thus S would have too many 6-secants. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_S(H+E_1) \longrightarrow \mathcal{O}_S(2H) \longrightarrow \mathcal{O}_D(2H) \longrightarrow 0.$$

From the above discussion $\mathcal{O}_D(2H) = \omega_D$ and the natural restriction map $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(2)) = \mathrm{H}^0(\mathcal{O}_S(2H)) \longrightarrow \mathrm{H}^0(\mathcal{O}_D(2H))$ is surjective, so taking cohomology of the exact sequence we see that $\mathrm{h}^1(\mathcal{O}_S(H+E_1)) = 1$. Now the natural multiplication maps

$$\mathrm{H}^1(\mathcal{O}_S(1)) \xrightarrow{\cdot H} \mathrm{H}^1(\mathcal{O}_S(2)),$$

where H is a hyperplane section containing E_1, factorize obviously

$$\mathrm{H}^1(\mathcal{O}_S(1)) \xrightarrow{\cdot E_1} \mathrm{H}^1(\mathcal{O}_S(H+E_1)) \xrightarrow{\cdot (H-E_1)} \mathrm{H}^1(\mathcal{O}_S(2)),$$

thus the plane P of the exceptional conic E_1 must coincide with the base locus of the pencil of hyperplanes parametrized by the directrix l of W. The planes which are base loci of the pencils parametrized by rulings of W build a non-normal cubic hypersurface $V \subset \mathbb{P}^4$ having P as double plane. V is the dual variety of $W \subset \check{\mathbb{P}}^4$.
The general curve D is a complete intersection $(3,3)$, hence it cuts the plane P along a scheme of length 9. Since $DE_1 = 3$ and $D^2 = 6$ it follows that P cuts S along E_1 and a scheme T of length 6 which is the base locus of $|D|$. Now a self-duality argument implies that T is the set of mutual intersection points of 4 lines l_i, $i = \overline{1,4}$ in the plane P, such that no three of them have common intersection points. It is easy to see now that for each line l_i, $i = \overline{1,4}$, there is a plane P_i in the ruling of V such that $P \cap P_i = l_i$. The hyperplane H_i, spanned by P and P_i, is special with respect to the pencil $|D|$, in the sense that the two cubics containing the member $D_i \in |D|$, lying in H_i, have P_i as component. Thus D_i splits as the union of a plane quintic curve C_i in the plane P_i and an elliptic normal curve of degree 4. In particular, the union $P \cup P_1 \cup P_2 \cup P_3 \cup P_4$ is part of the variety of 5-secants to S, and thus it is contained in the intersection of all quartic hypersurfaces containing S. The claim of the proposition follows now by degree reasons.□

Remark 3.40. *The cubic hypersurface $V \subset \mathbb{P}^4$, in the proof of proposition* 3.39, *intersects the surface $S \subset \mathbb{P}^4$ along the nine exceptional lines, twice the exceptional conic and the union of the four plane quintic curves $P_i \cap S$, $i = \overline{1,4}$.*□

Proposition (Ranestad) 3.41. *Let $P \subset \mathbb{P}^4$ be a plane and let $P_i \subset \mathbb{P}^4$ be general planes cutting P along four general lines l_1, l_2, l_3, l_4. Then $X_5 = P \cup P_1 \cup P_2 \cup P_3 \cup P_4$ can be linked* $(4,4)$ *to a smooth $K3$ surface $S \subset \mathbb{P}^4$, with $d = 11$, $\pi = 12$.*

Proof. We denote by $\{p_{ij}\} = l_i \cap l_j$, $1 \leq i < j \leq 4$, the mutual intersection points. One sees easily that X_5 is a local complete intersection scheme except for the points p_{ij}, where it is only Cohen-Macaulay. Therefore the proposition will follow via the liaison argument in (0.31) once we show that X_5 is cut out by quartic hypersurfaces. To see this, one proceeds by induction using the residual intersection sequences

$$0 \longrightarrow \mathcal{I}_{X_{k-1}}(m-1) \longrightarrow \mathcal{I}_{X_k}(m) \longrightarrow \mathcal{I}_{X_k \cap H_k, H_k}(m) \longrightarrow 0, \qquad m \in \mathbb{Z},$$

where $X_k = P \cup \bigcup_{i=1}^k P_i$ and H_k is a general hyperplane through the plane P_k. We obtain namely, that $\mathcal{I}_{X_k}(k-1)$ is globally generated and $\mathrm{h}^1(\mathcal{I}_{X_k}(k-1)) = 0$ for all $k = \overline{1,5}$. Since $\deg X_5 = 5$, $\pi(X_5) = 0$ and $\chi(X_5) = 1$ the linked surface has invariants $d = 11$, $\pi = 12$ and $\chi = 2$ and is therefore, by (3.35), a non-minimal $K3$ surface. □

Proposition 3.42. *Let S be a smooth surface in $\mathbb{P}^4$ with $d = 11$, $\pi = 12$, $\chi = 3$. Then S is either*
a) a regular, proper elliptic surface embedded by

$$H = H_{min} - 2E_1 - \sum_{i=2}^{4} E_i,$$

or
b) a regular, proper elliptic surface embedded by

$$H = H_{min} - \sum_{i=1}^{4} E_i,$$

or
c) a blown-up, general type Horikawa surface embedded by

$$H = H_{min} - \sum_{i=1}^{5} E_i.$$

Proof. In this case $K^2 = -4$ and the image of S through the adjunction morphism is a surface $S_1 \subset \mathbb{P}^{13}$ with invariants

$$d_1 = 29, \quad H_1K_1 = 7, \quad \pi_1 = 19, \quad K_1^2 = -4 + a$$

where $a \geq 0$ is the number of (-1) lines on S. In particular, Hodge index gives $a \leq 5$. Assume first that S is a surface of general type. Then $K_{\min}^2 \geq 1$, so there are at least 5 exceptional curves on S, say $E_1, \ldots, E_k$, for some $k \geq 5$. Now a curve $D_i \in |K_{\min} - E_i|$ has degree $\leq 11 - 5 - 2(k-5) - 1 = 15 - 2k$ and arithmetic genus $K_{\min}^2 + 1 \geq k - 3$, so it is easily seen that the only possibility is that $k = 5$ and all of the E_i's are exceptional lines. Lemma 0.39 shows further that S is in this case regular, and thus a surface of type *c*). If S is not of general type, then it is a regular proper elliptic surface by lemma 0.38, and in particular $p_g = 2$. A similar argument as above shows now that there are at least 3 exceptional lines on S. Therefore, either S is a surface of type *b*), or there exist only 3 exceptional lines on the surface and necessarily one more exceptional conic, and thus S is of type b).$\square$

(3.43.) Constructions. We show in the sequel that types a) and c) in the above list of surfaces with $d = 11$, $\pi = 12$, $\chi = 3$ exist. Lemma 3.36, together with the general results (1.6) and (1.7), provide the following cohomology table for the ideal sheaf of $S \subset \mathbb{P}^4$

i

2				
	2	1		
			1	a
				a+1

p

$\mathrm{h}^i(\mathcal{I}_S(p))$

Assume first that the cohomology table is minimal, i.e., $a = 0$. Then Beilinson's spectral sequence produces a vector bundle resolution of the ideal sheaf of a surface S_1

$$0 \longrightarrow 2\mathcal{O}(-1) \oplus 2\Omega^3(3) \xrightarrow{\varphi} \Omega^2(2) \oplus \Omega^1(1) \oplus \mathcal{O} \longrightarrow \mathcal{I}_{S_1}(4) \longrightarrow 0,$$

and one checks, via [Mac], that the degeneracy locus of a general morphism φ is a smooth surface in $\mathbb{P}^4$ with the desired invariants and a minimal free resolution

$$0 \leftarrow \mathcal{I}_{S_1} \longleftarrow \begin{matrix} \mathcal{O}(-4) \\ \oplus \\ 8\mathcal{O}(-5) \end{matrix} \nwarrow 13\mathcal{O}(-6) \leftarrow 6\mathcal{O}(-7) \leftarrow \mathcal{O}(-8) \leftarrow 0$$

In particular, I_{S_1} is generated by quintic hypersurfaces and thus S_1 has no 6-secant lines. From Le Barz's formula, which gives $N_6(11, 12, 3) = 3$, we deduce that there are 3 exceptional lines on the constructed surface, hence S_1 is a proper elliptic surface of type a). The determinantal construction suggests also the equivalent liaison construction in proposition 3.44 below.

We construct now a family of surfaces with non-minimal cohomology

i

2				
	2	1		
			1	1
				2

p

$\mathrm{h}^i(\mathcal{I}_S(p))$

Beilinson's theorem suggests this time to take

$$\mathcal{E} = 2\mathcal{O}(-1) \oplus 2\Omega^3(3) \qquad \text{and} \qquad \mathcal{F} = 2\mathcal{O} \oplus \ker(\Omega^2(2) \oplus \Omega^1(1) \xrightarrow{\psi} \mathcal{O}),$$

where ψ is a suitable epimorphism. We take ψ such that the component on $\Omega^2(2)$ is given by a decomposable element in the exterior algebra, i.e., without loss of generality, we set $\psi = (e_1 \wedge e_2, e_0)$, where $\mathbb{P}^4 = \mathbb{P}(\operatorname{span}_k(e_0, \ldots, e_4))$. One checks, via [Mac], that the degeneracy locus of a general morphism $\varphi \in \operatorname{Hom}(\mathcal{E}, \mathcal{F})$ is a smooth surface $S_2 \subset \mathbb{P}^4$, with syzygies

$$0 \leftarrow \mathcal{I}_{S_2} \longleftarrow \begin{array}{c} 2\mathcal{O}(-4)\\ \oplus \\ 3\mathcal{O}(-5)\\ \oplus \\ 2\mathcal{O}(-6)\end{array} \longleftarrow \begin{array}{c} 5\mathcal{O}(-6)\\ \oplus \\ 5\mathcal{O}(-7)\end{array} \longleftarrow \begin{array}{c} \mathcal{O}(-7)\\ \oplus \\ 4\mathcal{O}(-8)\end{array} \longleftarrow \mathcal{O}(-9) \leftarrow 0$$

One checks further that the distinguished plane $\Pi = \operatorname{span}_k(e_0, e_1, e_2)$ meets the constructed surface along a plane quintic curve with an embedded point at $p = \mathbb{P}(ke_0)$, thus S_2 has an infinity of 6-secant lines, namely all lines in Π going through p. By computing the base locus of the canonical pencil, we find out that there are this time 5 exceptional lines on the surface, thus S_2 is a blown-up Horikawa surface of type c). We remark that S_2 can be linked in the complete intersection of two quartic hypersurfaces to a scheme Z of degree 5 and sectional genus 0, which decomposes as $Z = 2\Pi \cup Q$, where 2Π is a double structure on the plane Π and Q is a rational cubic scroll.

Proposition 3.44. *A smooth rational surface $X \subset \mathbb{P}^4$ of degree 9, sectional genus 7 can be linked in the complete intersection of a quartic and a quintic hypersurfaces to a smooth, non-minimal proper elliptic surface $S \subset \mathbb{P}^4$, with $d = 11$, $\pi = 12$, $\chi = 3$, which is embedded by*

$$H = H_{\min} - 2E_0 - \sum_{i=1}^{3} E_i.$$

Proof. Smooth rational surfaces $X \subset \mathbb{P}^4$, with $d = 9$ and $\pi = 7$ have been studied in [Al2] and [AR]. They lie on a net of quartic hypersurfaces and on six extra independent quintic hypersurfaces, so they can be linked $(4, 5)$ to locally Cohen-Macaulay schemes $S \subset \mathbb{P}^4$ with invariants $d = 11$, $\pi = 12$, $\chi = 3$. We use the intrinsic description of the linear system of the embedding $X \subset \mathbb{P}^4$, which is due to J. Alexander [Al2], to describe this liaison. We'll also need and recall along the proof the results in [AR]. To begin with, it follows from [Al2] that X is $\mathbb{P}^2$ blown-up in 15 points and

$$H_X \equiv 9l - \sum_{i=1}^{6} 3F_i - \sum_{j=7}^{9} 2F_j - \sum_{k=10}^{15} F_k,$$

where F_i, $i = \overline{1, 15}$ are the exceptional curves of the blow-up map, and where the 15 points are chosen such that there exists a pencil

$$|D| = |6l - \sum_{i=1}^{6} 2F_i - \sum_{j=7}^{9} F_j - \sum_{k=10}^{15} F_k|$$

with base points on F_7, F_8 and F_9. Residual to the pencil there is a plane cubic curve $H_X - D \equiv 3l - \sum_{i=1}^{9} F_i$, and we'll denote in the sequel with Π its plane. Since the base points of the pencil $|D|$ are on the lines F_i, $i = \overline{7,9}$, there exist on X three plane quartic curves

$$D_m \equiv 6l - \sum_{i=1}^{6} 2F_i - \sum_{j=7}^{9} F_j - \sum_{k=10}^{15} F_k - F_{m+6},$$

for $m = \overline{1,3}$. Let Π_1, Π_2, Π_3 be the planes of these curves. Each pencil $|H_X - D_m|$, $m = \overline{1,3}$, has a base point p_m in the corresponding plane Π_m, thus any line in Π_m through p_m is a 5-secant to X, and it follows that any quartic containing X must also contain the scheme $\Pi_1 \cup \Pi_2 \cup \Pi_3$. Now $D_m(H_X - D) = 2$ for all m, so each Π_m meets Π along a line, and thus Π is also contained in all quartics containing X. The liaison result in [AR] says that the complete intersection of two general quartics containing X decomposes as $\Pi \cup \Pi_1 \cup \Pi_2 \cup \Pi_3 \cup T$, where T is a cubic Del Pezzo surface intersecting Π_i, $i = \overline{1,3}$ along lines and X along the hyperplane section cut on X by the hyperplane spanned by the Del Pezzo surface.

We fix now a general quartic hypersurface containing X, and thus also the union $\Pi \cup \Pi_1 \cup \Pi_2 \cup \Pi_3$. A Bertini argument shows that we can link X on V, via a quintic hypersurface W, to a smooth surface S. Now W cuts each of the planes Π_i, $i = \overline{1,3}$, along the quartic D_i and an extra line, call it E_i, which then necessarily lies on S. The same argument shows that Π cuts the surface S along a conic, which we'll denote in the sequel by E_0. On another side, the above discussion shows that there exists on V a pencil of Del Pezzo surfaces $T_{(\lambda:\mu)}$, all meeting X along hyperplane sections $H_{(\lambda:\mu)}$ of it. Take now the embedding $H_{T_{(\lambda:\mu)}} = 3l - \sum_{i=1}^{6} G_i$ of $T_{(\lambda:\mu)}$ such that each Π_i intersects $T_{(\lambda:\mu)}$ along the line in the class G_i, for all $i = \overline{1,3}$. Then the curve $H_{(\lambda:\mu)}$ has numerical class $H_{(\lambda:\mu)} \equiv 4H_{T_{(\lambda:\mu)}} - \sum_{i=1}^{3} G_i \equiv 12l - \sum_{i=1}^{3} 5G_i - \sum_{j=4}^{6} 4G_j$ and thus the pencil $T_{(\lambda:\mu)}$ will cut on S a pencil of elliptic curves $M_{(\lambda:\mu)}$ of degree 6, in the class $3l - \sum_{j=4}^{6} G_j$. In particular, it follows that S is a proper elliptic surface. More precisely, the residual curve $C \equiv H - (5H_{T_{(\lambda:\mu)}} - H_{(\lambda:\mu)})$ of a member in the pencil has degree 5 and arithmetic genus $p_a(C) = M_{(\lambda:\mu)}^2 + 6$, and since $M_{(\lambda:\mu)}^2 \geq 0$ we deduce that $M_{(\lambda:\mu)}^2 = 0$ and C is a plane curve. It follows that $|M_{(\lambda:\mu)}|$ coincides with the residual pencil of the plane quintic curve C, and moreover that $|M_{(\lambda:\mu)}| = |K_{\min}|$. In particular the exceptional part of the canonical divisor on S has degree 5, and thus S is an elliptic surface of type *a*) in the notation of proposition 3.42. Furthermore, it is easily checked that E_1, E_2 and E_3 are the exceptional lines of S, while E_0 is the exceptional conic.□

Proposition 3.45. *Let S be a smooth surface in $\mathbb{P}^4$ with $d = 11$, $\pi = 12$, $\chi = 4$. Then S is either*
a) a blown-up, general type Horikawa surface embedded by

$$H = H_{min} - E_1,$$

and having no 6-secant lines, or

b) a minimal, regular, general type surface with only one 6-secant line, or
c) a regular, non-minimal, general type surface embedded by

$$H = H_{min} - E_1 - E_2,$$

and having infinitely many 6-secant lines, namely all lines in one of the rulings of a smooth quadric surface.

Proof. The double point formula gives $K^2 = 2$, thus S is of general type. Furthermore S is regular. Otherwise, the inequality in [Deb, Th.6.1] reads

$$K_{min}^2 \geq 2p_g \geq 8$$

so S would have at least 6 exceptional curves, whence $HK_{min} \leq 5$ while $p_a(K_{min}) = K_{min}^2 + 1 \geq 9$, which is absurd. Riemann-Roch and lemma 3.36 yield then the following cohomology table for the ideal sheaf of S

i

3				
	1			
			2	a
				a

p

$\mathrm{h}^i(\mathcal{I}_S(p))$

.

We'll discuss in the sequel the possible values for a and their corresponding linear systems. Let $\Delta(4)$ be the locus in $\check{\mathbb{P}}^4$ where $(\mathrm{h}^0(\mathcal{I}_H(4))) \cdot (\mathrm{h}^1(\mathcal{I}_H(4))) \neq 0$.

Lemma 3.46. *If S has a 6-secant, then $\Delta(4)$ contains a plane.*

Proof. Let π be a general plane through the 6-secant. Then $\mathrm{h}^1(\mathcal{I}_{\pi \cap S}(4)) > 0$ and since $\mathrm{h}^1(\mathcal{O}_H(3)) = \mathrm{h}^0(\mathcal{O}_H(K - 2H)) = 0$ for the general hyperplane containing $\pi \cap S$, it follows that $\mathrm{h}^1(\mathcal{I}_H(4)) > 0$ and thus the claim of the lemma. □

Lemma 3.47. *Any hyperplane section $H \in \Delta(4)$ has a proper 6-secant or contains a plane curve as component.*

Proof. Assume that H has no plane curve as component. Then the sequence

$$0 \longrightarrow \mathcal{I}_H(3) \xrightarrow{H'} \mathcal{I}_H(4) \longrightarrow \mathcal{I}_{H \cap H'}(4) \longrightarrow 0$$

is exact for all hyperplane sections H'. Now $\mathrm{h}^1(\mathcal{I}_H(3)) = \mathrm{h}^1(\mathcal{I}_S(3)) = 2$ so taking cohomology of the exact sequence we see that there is at least one plane section $\pi \cap S$ in H for which $\mathrm{h}^1(\mathcal{I}_{\pi \cap S}(4)) > 0$. This plane section is a scheme of length 11. If it contains a subscheme of length 10 which is contained in a conic then $h^0(\mathcal{I}_{\pi\cap S}(3)) > 1$. On the other hand, from the above cohomology table $h^0(\mathcal{I}_H(3)) = 0$, hence taking cohomology of the exact sequence

$$0 \longrightarrow \mathcal{I}_H(2) \longrightarrow \mathcal{I}_H(3) \longrightarrow \mathcal{I}_{\pi\cap S}(3) \longrightarrow 0$$

we see that $h^1(\mathcal{I}_H(2)) \geq 2$. But, since $h^2(\mathcal{I}_S(1)) = 1$ this implies that $h^1(\mathcal{I}_S(2)) > 0$, which is a contradiction. Therefore, it follows from lemma 0.41 that $\pi \cap S$ contains a subscheme of length 6 which is contained in a line, and the lemma follows.□

Lemma 3.48. *S has only finitely many plane curves.*

Proof. Since any pencil of plane curves on S is linear the residual curve in a hyperplane section containing a general element of the family is again a plane curve. But then the hyperplane section is contained in a quadric and this contradicts Severi's theorem.□

If $a \geq 3$ then $\Delta(4) = \check{\mathbb{P}}^4$ and the above lemmas show that the general hyperplane section of S has 6-secants, while S is contained in at least a net of irreducible quartics, a contradiction. Thus we are left to consider the cases where $a \in \{0, 1, 2\}$.

When $a = 0$ we easily see that $\Delta(4) = \emptyset$, hence S has no 6-secant lines by lemma 3.46. On the other hand, Le Barz's formula (0.11) gives $N_6 = 1$, so S has exactly one exceptional line E_1. If there exists also another exceptional curve E_2 on S then $K_{min}^2 \geq 4$, $HK_{min} \leq 8$, thus a curve $D \in |K_{min} - E_1 - E_2|$ would have degree at most 5 and arithmetic genus at least 5. Since in any case $p_a(D) \neq 6$, this is a contradiction by lemma 0.34 and therefore S is embedded by $H = H_{min} - E_1$.

In case $a = 1$ the above lemmas imply that $\Delta(4) = \mathbb{P}^2$ and that S has only one 6-secant line. Therefore there are no exceptional lines on S this time. In fact S is minimal. The adjunction mapping process and a reasoning similar to the previous case show that the only other alternative is that there exists an exceptional conic E on S. But then a general curve $D \in |H - E|$ is irreducible, of degree 9 and arithmetic genus 10 and, since $h^2(\mathcal{O}_S(2H - D)) = h^0(\mathcal{O}_S(K - H - E)) = 0$, the cohomology of the sequence

$$0 \longrightarrow \mathcal{O}_S(2H - D) \longrightarrow \mathcal{O}_S(2H) \longrightarrow \mathcal{O}_D(2H) \longrightarrow 0$$

shows that $h^1(\mathcal{O}_D(2)) = 0$, whence finally by Riemann-Roch that D lies on a quadric. It is necessarily a curve of type $(3, 6)$ on a smooth quadric, so S would have too many 6-secants, which is absurd.

In the last case $a = 2$ and $\Delta(4)$ is a determinantal hyperquadric in $\check{\mathbb{P}}^4$. The above lemmas imply that all lines in one of the rulings of the dual quadric surface Q are 6-secants to S. Moreover, if π is a general plane spanned by two concurrent rulings of the quadric Q, then $h^0(\mathcal{I}_{\pi\cap S}(3)) = 1$ and $h^0(\mathcal{I}_{\pi\cap S}(4)) = 5$. It follows that π meets S in at most two points outside the two chosen rulings, and therefore the quadric Q cuts S along a curve D of type $(6, a)$, with $a \geq 3$. On another side, if $a = 4$, then the residual curve $H - D$ is a line and thus $p_a(H - D) = 0$ yields $D^2 = 24$, which would contradict Hodge index. We conclude that Q meets S along a curve D of type $(6, 3)$. Consider now again the residual curve $H - D$. It has degree 2 and arithmetic genus $D^2 - 6$, and Bezout's theorem gives

$D(H-D) \leq 4$, i.e., $D^2 \geq 5$, so either equality holds and $H-D$ is the union of two skew lines, or $D^2 = 6$ and $H-D$ is a conic. The last case cannot occur, since otherwise the 6-secants of S would fill up a 3-fold, which is impossible for a surface contained in a pencil of irreducible quartics. Therefore $H-D = E_1 + E_2$, with $HE_1 = HE_2 = 1$ and $E_1E_2 = 0$, and since $E_1^2 + E_2^2 = (H-D)^2 = -2$, while $E_i^2 \leq -1$, $i = 1, 2$, we deduce that E_1 and E_2 are exceptional lines on S. An argument similar to that used in case *a*) shows that there are no further exceptional curves on S, whence S is embedded by $H = H_{\min} - E_1 - E_2$.$\square$

(3.49.) Constructions. We'll show in the sequel that all types of surfaces in proposition 3.45 exist. For convenience, we denote by S_0, S_1 and S_2 surfaces corresponding to types *a*), *b*) and respectively *c*). Then Beilinson's spectral sequence and the above discussion imply that the ideal sheaf of any of the surfaces has a resolution of type

$$0 \longrightarrow 3\mathcal{O}(-1) \oplus \Omega^3(3) \xrightarrow{\varphi_a} \mathcal{G}_a \oplus a\mathcal{O} \longrightarrow \mathcal{I}_{S_a}(4) \longrightarrow 0 \qquad (*)$$

where $a \in \{0, 1, 2\}$, and $\mathcal{G}_a$ is a vector bundle, kernel of a morphism

$$0 \longrightarrow \mathcal{G}_a \longrightarrow 2\Omega^1(1) \xrightarrow{\psi_a} a\mathcal{O} \longrightarrow 0.$$

Also $\Delta(4)$ coincides with the locus in $\check{\mathbb{P}}^4$ where the $2 \times a$ matrix defining ψ_a is dropping rank. Vice-versa, we use the information on $\Delta(4)$ to construct such surfaces, and take $\mathcal{G}_0 = 2\Omega^1(1)$ and as $\mathcal{G}_a$, $a = 1, 2$, the kernel of a suitable epimorphism $\psi_a : 2\Omega^1(1) \to a\mathcal{O}$ such that the $2 \times a$ matrix M_a defining it degenerates on a plane if $a = 1$, or on a point quadric cone if $a = 2$. Namely, without loss of generality, one can take $M_1 = (e_1, e_2)$ and $M_2 = \begin{pmatrix} e_3 & e_4 \\ e_0 & e_1 \end{pmatrix}$, and define S_a as the degeneracy locus of a general morphism φ_a as above in $(*)$. Smoothness can be checked in examples with [Mac]. From $(*)$ we obtain also the following minimal free resolutions of the ideal sheaf:

$$0 \leftarrow \mathcal{I}_{S_0} \leftarrow 12\mathcal{O}(-5) \longleftarrow 19\mathcal{O}(-6) \longleftarrow 10\mathcal{O}(-7) \longleftarrow 2\mathcal{O}(-8) \leftarrow 0$$

thus S_0 is cut out by quintics and has no 6-secants,

$$\begin{array}{ccccccccc} & & \mathcal{O}(-4) & & & & & & \\ & & \oplus & & & & & & \\ 0 \leftarrow \mathcal{I}_{S_1} & \leftarrow & 7\mathcal{O}(-5) & \nwarrow & 10\mathcal{O}(-6) & & 3\mathcal{O}(-7) & & \\ & & \oplus & & \oplus & \leftarrow & \oplus & \nwarrow & \\ & & \mathcal{O}(-6) & & 3\mathcal{O}(-7) & & 3\mathcal{O}(-8) & & \mathcal{O}(-9) \leftarrow 0 \end{array}$$

where the linear part of the last syzygy are the equations of the unique 6-secant to S_1, and

$$\begin{array}{ccccccc} & & 2\mathcal{O}(-4) & & & & \\ & & \oplus & & & & \\ 0 \leftarrow \mathcal{I}_{S_2} & \leftarrow & 2\mathcal{O}(-5) & \nwarrow & 3\mathcal{O}(-6) & & \\ & & \oplus & & \oplus & \nwarrow & \\ & & 4\mathcal{O}(-6) & & 10\mathcal{O}(-7) & & 8\mathcal{O}(-8) \longleftarrow 2\mathcal{O}(-9) \leftarrow 0\,. \end{array}$$

The quartics and the quintics in the ideal I_{S_2} cut out the union of S_2 with the quadric of 6-secant lines.

Remark 3.50. *A surface $S \subset \mathbb{P}^4$, with $d = 11$, $\pi = 12$, $\chi = 4$, of type c) in the above proposition, can be linked $(4,4)$ to a scheme Z of degree 5, sectional genus 0, which is the union of a double structure on a quadric and a plane which meets the quadric along one of its rulings.*

Proposition 3.51. *A smooth surface $S \subset \mathbb{P}^4$, with $d = 11$, $\pi = 12$, $\chi = 5$ is a regular, minimal surface of general type.*

Proof. The same argument we used in proposition 3.45 shows that S is regular. Moreover, Severi's theorem and lemma 3.36. yield $\mathrm{h}^1(\mathcal{O}_S(k)) = 0$, for all $k \geq 1$, thus combining with (1.6) and (1.7) we obtain also $\mathrm{h}^1(\mathcal{I}_S(2)) = 1$ and $\mathrm{h}^1(\mathcal{I}_S(3)) = 3$. It follows that $\mathrm{h}^1(\mathcal{I}_H(3)) = 3$ for some hyperplane sections H, or equivalently $\mathrm{h}^0(\mathcal{I}_H(3)) = 1$, whence $\mathrm{h}^0(\mathcal{I}_H(4)) \geq 4$, and furthermore $\mathrm{h}^0(\mathcal{I}_S(4)) \geq 1$. Assume that S is not minimal and let E be an exceptional curve on it. Then a curve $D \in |K_{\min} - 2E|$ will have degree $\leq HK - 3HE \leq 8$ and arithmetic genus $p_a(D) = K_{\min}^2 + 1 \geq 10$. Then by lemma 0.34 such a curve would have as component a plane curve of degree at least 6 which is impossible by lemma 0.36. Therefore S is a minimal surface as claimed.□

(3.52.) Construction. As a consequence of the above discussion the ideal sheaf of the surface S has the following cohomology table

i

5				
		1	3	2
				1

p

$\mathrm{h}^i(\mathcal{I}_S(p))$

since if $\mathrm{h}^1(\mathcal{I}_S(4)) = 3$, then $\mathrm{h}^0(\mathcal{I}_S(4)) = 2$ and Beilinson's spectral sequence implies that the two quartics in I_S would have non-trivial linear syzygies, which is impossible if the surface is smooth. Similar arguments, together with the information in the proof of (3.51) allows to exclude the cases when $\mathrm{h}^1(\mathcal{I}_S(4)) \geq 4$. Beilinson's theorem suggests to take

$$\mathcal{E} = 5\mathcal{O}(-1) \qquad \text{and} \qquad \mathcal{F} = Syz_1(\mathrm{H}^1(\mathcal{I}_S(*+4))),$$

and since a monogenous artinian module with Hilbert function $(1,3,2)$ is the tensor product of a similar module on 3 variables with the Koszul complex of a plane, we have without loss of generality

$$\mathcal{F} = \ker(4\mathcal{O}(-1) \oplus \mathcal{O} \xrightarrow{\psi} \mathcal{O}(1)),$$

with $\psi = (x_0, x_1, q_1, \ldots, q_4)$, where $q_i \in k[x_2, x_3, x_4]$ are general quadrics. The degeneracy locus of a general morphism $\varphi \in \mathrm{Hom}(\mathcal{E}, \mathcal{F})$ is a smooth surface $S \subset \mathbb{P}^4$ with the desired invariants and syzygies

$$\begin{array}{ccccccccc} & & \mathcal{O}(-4) & & & & & & \\ & & \oplus & & & & & & \\ 0 \leftarrow \mathcal{I}_S \longleftarrow & & 6\mathcal{O}(-5) & \nwarrow & 8\mathcal{O}(-6) & & 2\mathcal{O}(-7) & & \\ & & \oplus & & \oplus & \longleftarrow & \oplus & \nwarrow & \\ & & 3\mathcal{O}(-6) & & 8\mathcal{O}(-7) & & 7\mathcal{O}(-8) & & 2\mathcal{O}(-9) \leftarrow 0 \end{array}$$

The quintics in the ideal I_S cut out the surface and an extra plane Π. The plane Π meets the surface along a plane quartic curve and in three other points, thus S has three 6-secant lines, namely the three lines joining pairwise these points. Since S is minimal, this matches Le Barz's formula which gives $N_6(11, 12, 6) = 3$.

VI. Surfaces with $d = 11$, $\pi = 13$

The double point formula gives $K^2 = 6\chi - 27$, so Hodge index implies $\chi \leq 7$. On the other side, Lemma 0.20 yields $\chi \geq 1$. Also, when $\kappa(S) \geq 0$, the pseudo-effectiveness of K gives via (0.15) $\chi \geq 3$, whence in particular $\kappa(S) \geq 1$.

Proposition 3.53. *There are no smooth, rational surfaces $S \subset \mathbb{P}^4$ with invariants $d = 11$, $\pi = 13$.*

Proof. We use the adjunction mapping to exclude this case. Theorem 0.13 gives for the surface $S_1 = \varphi_{H+K}(S) \subset \mathbb{P}^{12}$

$$d_1 = 16, \quad H_1K_1 = -8, \quad \pi_1 = 5, \quad K_1^2 = -21 + a$$

where $a \geq 0$ is the number of (-1) lines on S. Now (0.13) applied for S_1 implies that $(H_1 + K_1)^2 = -21 + a \geq 0$. If $a > 21$ then $S_2 = \varphi_{H_1+K_1}(S_1)$ is a surface in $\mathbb{P}^4$ with invariants

$$d_2 = -21 + a, \quad H_2K_2 = -29 + a, \quad \pi_2 = -26 + a$$

whence necessarily $a \geq 26$. But this is impossible since by Hodge index $K_1^2 = -21 + a \leq \frac{64}{16} = 4$. We are left with the case $a = 21$ which means that S_1 is a conic bundle having, since $K_1^2 = 0$, eight singular fibres in the ruling. Let $\mathbb{F}_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$, $e \geq 0$ be the relative minimal model of S_1, C_0 a section with minimal negative self-intersection $-e$ and f a ruling. Thus we may write, pulling back C_0 and f

$$H \equiv 4C_0 + (2e + 8)f - \sum_{i=1}^{8} 2E_i - \sum_{j=9}^{29} E_j$$

Let now $B \equiv f - E_j$, for some $j \geq 9$. Then $HB = 3$, so B is contained in a hyperplane section of S. Let C be the residual curve. It has degree 8 and arithmetic genus 10, hence by Lemma 0.34 contains a plane sextic curve as component. But this is a contradiction since $\mathrm{h}^1(\mathcal{O}_S(1)) = 5$ and hence, by Lemma 0.36, the maximal degree of a plane curve on S is 5.$\square$

Proposition 3.54. *There are no smooth surfaces $S \subset \mathbb{P}^4$ with $d = 11$, $\pi = 13$, $\chi = 3$.*

Proof. In this case $K^2 = -9$ and the image of S through the adjunction morphism is a surface $S_1 = \varphi_{H+K}(S) \subset \mathbb{P}^{14}$ with invariants

$$d_1 = 28, \quad H_1K_1 = 4, \quad \pi_1 = 17, \quad \text{and} \quad K_1^2 = -9 + a$$

where $a \geq 0$ is the number of (-1) lines on S. Now $H_1^2 > (H_1K_1)^2$, so by lemma 0.18 S is a proper elliptic surface. Moreover, since $H_1K_1 = 4$, Kodaira's formula for the canonical class of an elliptic fibration [BPV] implies that $p_g = 2$ and that S is either minimal, or has exactly one (-1) line.
In the first case the moving part of $|K|$ is a pencil of elliptic quartic curves and hence the residual of a general member in the hyperplane section it spans will be a curve D of

degree 7 and arithmetic genus 9. By Lemma 0.34 such a curve has a plane sextic curve as component and since $h^1(\mathcal{O}_S(1)) = 3$ this contradicts the bound in (0.36).
In the second case the elliptic fibration of S is by plane cubic curves and the residual of a fibre in a hyperplane section containing it will have degree 8 and arithmetic genus 10. Again, by Lemma 0.34, such a curve has a plane sextic as component contradicting the results of (0.36) □

Proposition 3.55. *There are no smooth surfaces $S \subset \mathbb{P}^4$ with $d = 11$, $\pi = 13$, $\chi = 4$.*

Proof. In the above hypothesis $K^2 = -3$. Let S_{min} be the minimal model of S. If S is of general type, then by Noether's inequality [BPV]

$$K^2_{min} \geq 2p_g - 4 \geq 4$$

Thus S has at least 5 exceptional curves E_i, $i = \overline{1,5}$ and $HK_{min} \leq 13 - 5 = 8$ and $p_a(K_{min}) \geq 3$. Since $p_g \geq 3$ we can find a curve $C \in |K_{min} - E_1 - E_2|$. It has degree $HC \leq 6$ and arithmetic genus $p_a(C) \geq 3$. If one of E_1 or E_2 has degree at least 2 then $HC \leq 4$ and $p_a(C) \geq 3$ which is absurd in view of Lemma 0.34. Therefore S has only (-1) lines and it is easily seen that there are two possibilities, either 5 or 6 exceptional lines. Nevertheless, in both cases, arguments as above or as in (0.38) give $p_g = 3$ and $q = 0$.
In the first case we compute $K^2_{min} = 2$, whence $HC = 6$, $p_a(C) = 3$. If C is irreducible then it spans only a hyperplane and the residual curve will have degree 5 and arithmetic genus 5 which is impossible by Lemma 0.34. It remains that C decomposes as the union of a plane quintic curve A and two skew lines L_1 and L_2 or as the union of a plane quartic curve A and a conic Q such that each line (resp. the conic) meets A in one point. Neither of the lines L_1 or L_2 is exceptional since, e.g., from $L_1 = E_1$ and $CE_1 = (K_{min} - E_1 - E_2)E_1 = 1$ would follow $AL_1 = AE_1 = 2$. Therefore $L_i^2 \leq -2$, $i = 1, 2$ (resp. $Q^2 \leq -2$) whence $A^2 \geq 0$ because $C^2 = 0$. On the other side Hodge index gives $A^2 \leq 1$. The residual pencil $|H - A|$ has degree 7 and arithmetic genus $7 + A^2 \geq 7$ so, by Lemma 0.34, $A^2 = 0$ and each of its members decomposes into a plane quintic curve and a conic which meet along a subscheme of length two. This means that the plane spanned by A cuts the surface S along a curve of degree at least 6, thus contradicting the bound in (0.36). In the case of 6 exceptional lines we compute $K^2_{min} = 3$, $HC = 5$, $p_a(C) = 4$ which is again impossible by Lemma 0.34.
The last case we have to discuss is when S is a proper elliptic surface. Then K_{min} is numerically equivalent to $nF + \sum_{i=1}^{k} F_i$, $n \geq p_g - 1 \geq 2$, $k \geq 0$, where F is a fibre of the elliptic fibration and the F_i's are reduced parts of not necessarily distinct fibers. Since $HK_{min} \leq 13 - 3 = 10$ we get $HF \leq 5$.
If $HF = 5$ then $p_g = 2$, $k = 0$ and S has exactly 3 exceptional lines. A curve $D \equiv F - E_1$ has degree 4 and arithmetic genus 1, hence it spans only a $\mathbb{P}^3$. But then the residual curve $H - D$ will have degree 7 and arithmetic genus 8 which is impossible by lemma 0.34.
If $HF \leq 4$, since a curve $D \equiv F - E_1$ with E_1 exceptional curve has degree at least 3, it follows that $HF = 4$, $n = 2$, $k \geq 1$ and S has 3 exceptional lines. But then $HF_1 \leq 2$ which is absurd for a curve of arithmetic genus 1.□

Proposition 3.56. *A smooth surface $S \subset \mathbb{P}^4$ with $d = 11$, $\pi = 13$ and $\chi = 5$ is the degeneracy locus of a morphism φ inducing*

$$0 \longrightarrow 4\mathcal{O}(-1) \oplus \Omega^3(3) \xrightarrow{\varphi} \Omega^2(2) \oplus 3\mathcal{O} \longrightarrow \mathcal{I}_S(4) \longrightarrow 0.$$

Proof. To show regularity we proceed as in (3.45). Namely, when $q \geq 1$, the inequality in [Deb, Th.6.1] gives $K^2_{min} \geq 10$, hence $HK_{min} \leq 6$ while $p_a(K_{min}) \geq 11$, which is absurd. We determine next the cohomology table of S. First of all $\mathrm{h}^0(\mathcal{O}_S(K-H)) = 0$ by lemma 0.17, so Riemann-Roch gives $\mathrm{h}^1(\mathcal{O}_S(1)) = 1$. Assume now that $\mathrm{h}^1(\mathcal{O}_S(2)) \geq 2$. Then there exists a line $L \subset \mathbb{P}^4$ which is the base locus of a net of hyperplanes for which $\mathrm{h}^1(\mathcal{O}_H(2)) \geq 2$. The line L lies on S; otherwise the general H in the net is smooth, whence hyperelliptic by Clifford's theorem and this is impossible by theorem 0.13 and [E]. Therefore the general H in the net decomposes $H = L + C$, where C is a smooth curve with $HC = 10$ and $g(C) = 13 + L^2$. Moreover C is irreducible since otherwise $|H - L|$ would be composed with a pencil of plane curves and this leads to a contradiction as in the proof of lemma 3.36. Consider the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_C(2H - L) \longrightarrow \mathcal{O}_H(2H) \longrightarrow \mathcal{O}_L(2H) \longrightarrow 0.$$

$\mathrm{h}^1(\mathcal{O}_H(2H)) = 2$, whereas $\mathrm{h}^1(\mathcal{O}_L(2H)) = 0$, so $\mathrm{h}^0(\mathcal{O}_C(K-H)) = \mathrm{h}^1(\mathcal{O}_C(2H-L)) \geq 2$. Therefore, either $(K-H)C = 5 + L^2 \geq 3$, or $L^2 = -3$, $\mathrm{h}^1(\mathcal{O}_C(2H-L)) = 2$ and C is hyperelliptic. When $L^2 \geq -2$, it follows from the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(L^2+1) \longrightarrow \mathcal{O}_H(2H) \longrightarrow \mathcal{O}_C(2H) \longrightarrow 0$$

that $\mathrm{h}^0(\mathcal{O}_C(K-H-L)) = \mathrm{h}^1(\mathcal{O}_C(2H)) = 2$. This implies again that C is hyperelliptic and $L^2 = -1$, because $C(K-H-L) = 2L^2 + 4$ and, in any case, $L^2 \leq -1$. In both cases we obtained a contradiction, since a hyperelliptic curve C can never be embedded in a projective space with degree at most $g(C) + 2$ (see for instance [ACGH, p.221]). Therefore Riemann-Roch and (1.6) give $\mathrm{h}^1(\mathcal{O}_S(2)) = 1$ and thus also $\mathrm{h}^1(\mathcal{O}_S(k)) = 0$, for $k \geq 3$, since otherwise $\mathrm{h}^1(\mathcal{O}_H(3)) > 0$ for at least a web of hyperplane sections, which is absurd because the general H in the web is smooth and $\mathcal{O}_H(3H)$ is non special. From (1.7) it follows $\mathrm{h}^1(\mathcal{I}_S(3)) = 0$ and a similar argument to that used in the proof of (3.37) shows that $\mathrm{h}^1(\mathcal{I}_S(4)) = 0$. We obtain the following cohomology table

i

4				
	1	1		
				3

p

$\mathrm{h}^i(\mathcal{I}_S(p))$

and thus the claim of the proposition is a consequence of Beilinson's spectral sequence.□

Corollary 3.57. *A smooth surface $S \subset \mathbb{P}^4$ with $d = 11$, $\pi = 13$, $\chi = 5$ is a regular, non-minimal, general type surface with only one exceptional line.*

Proof. S has one exceptional line E_1 since $N_6 = 1$ by Le Barz's formula, and because it follows from the above proposition that there are no 6-secants. If E_2 is another exceptional curve, then for a curve $D \in |K_{min} - 2E_2|$ we would have $HD \leq 6$ and $p_a(D) \geq 5$, whence necessary E_2 is a (-1) conic, $K^2_{min} = 5$ and D decomposes, by (0.34), as $D = A + L$, where A is a plane quintic and L a line which doesn't meet A. But $L^2 \leq -1$ and $A^2 \leq 2$ by Hodge index, so $D^2 = A^2 + L^2 = 1$ implies that $L^2 = -1$ and thus that L coincides with E_1, which is absurd because $DE_1 = 0$.□

(3.58.) Construction. A surface $S \subset \mathbb{P}^4$ with the above invariants can be constructed as the degeneracy locus of a general morphism $\varphi \in \mathrm{Hom}(4\mathcal{O}(-1) \oplus \Omega^3(3), \Omega^2(2) \oplus 3\mathcal{O})$. In particular the ideal sheaf has a minimal free resolution of type

$$0 \leftarrow \mathcal{I}_S \longleftarrow \begin{array}{c} 3\mathcal{O}(-4) \\ \oplus \\ 2\mathcal{O}(-5) \end{array} \longleftarrow \begin{array}{c} \mathcal{O}(-5) \\ \oplus \\ 4\mathcal{O}(-6) \end{array} \nwarrow \mathcal{O}(-7) \leftarrow 0$$

where the three linear syzygies in the top row are the equations defining the unique exceptional line of S. An equivalent construction, which provides also a smoothness argument, is contained in the following

Proposition 3.59. *A smooth, general type surface $Y \subset \mathbb{P}^4$, with $d = 9$ and $\pi = 8$ can be linked in the complete intersection of a quartic and a quintic hypersurface to a smooth surface $S \subset \mathbb{P}^4$ with invariants $d = 11$, $\pi = 13$, $p_g = 4$ and $q = 0$.*

Proof. It follows from [AR] that Y is cut out by quartic hypersurfaces, so S is smooth for a general choice of the liaison. Formula (0.26) and the cohomology of the liaison exact sequence

$$0 \longrightarrow \mathcal{O}_S(K) \longrightarrow \mathcal{O}_\Sigma(4) \longrightarrow \mathcal{O}_Y(4) \longrightarrow 0,$$

where Σ is the complete intersection $(4, 5)$, give further $\pi(S) = 13$, $p_g(S) = 4$, $q(S) = 0$.□

Proposition 3.60. *A smooth surface $S \subset \mathbb{P}^4$ with $d = 11$, $\pi = 13$, $\chi = 6$ is a regular, minimal, general type surface, linked in the complete intersection of two quartic hypersurfaces to an elliptic quintic scroll.*

Proof. We have $K^2 = 9$, so S is of general type. A smooth curve of degree 11 and genus 13 in a projective space spans only a $\mathbb{P}^3$, thus $\mathrm{h}^0(\mathcal{O}_H(H)) = 4$. By Lemma 0.17 $\mathrm{h}^0(\mathcal{O}_S(K - H)) = 0$ so $\mathrm{h}^1(\mathcal{O}_S(1)) = 0$ and the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(H) \longrightarrow \mathcal{O}_H(H) \longrightarrow 0$$

give $q = \mathrm{h}^1(\mathcal{O}_S) = 0$, i.e., S is a regular surface.

We determine next the cohomology table of S. Since $\mathrm{h}^1(\mathcal{O}_H(2)) = \mathrm{h}^0(\mathcal{O}_H(K-H)) \leq 1$ (if non-zero, $\mathrm{h}^0(\mathcal{O}_H(K-H)) \leq H(K-H) = 2$ and in case of equality S would have hyperelliptic hyperplane sections, which is impossible by [So], [E]) the cohomology of the above exact sequence twisted by H yields $\mathrm{h}^1(\mathcal{O}_S(2)) \leq 1$. In any case $\mathrm{h}^1(\mathcal{O}_H(3)) = 0$ for all irreducible hyperplane sections H and thus $\mathrm{h}^1(\mathcal{O}_S(n)) \geq \mathrm{h}^1(\mathcal{O}_S(n+1))$ for all $n \geq 2$ and, in particular $\mathrm{h}^1(\mathcal{O}_S(3)) \leq 1$. But if $\mathrm{h}^1(\mathcal{O}_S(3)) > 0$, then $\mathrm{h}^1(\mathcal{O}_H(3)) > 0$ for at least a web of hyperplane sections, which is absurd, since the general H in the web is smooth, while $\mathcal{O}_H(3)$ is non-special. It follows that $\mathrm{h}^1(\mathcal{O}_S(k)) = 0$, for all $k \geq 3$.

S is not contained in a cubic hypersurface, because otherwise $\mathrm{h}^1(\mathcal{I}_S(3)) = \mathrm{h}^0(\mathcal{I}_S(3)) - \chi(\mathcal{I}_S(3)) \geq 2$, and this is impossible for a surface which, by theorem 1.7, is in the liaison class of the Veronese surface. Therefore $\mathrm{h}^0(\mathcal{I}_S(3)) = 0$ and $\mathrm{h}^1(\mathcal{I}_S(3)) = 1$.

Claim. $\mathrm{h}^1(\mathcal{O}_S(2)) = \mathrm{h}^1(\mathcal{I}_S(2)) = 0$.

Proof of the claim. Assume that this is not the case; i.e., $\mathrm{h}^1(\mathcal{O}_S(2)) = \mathrm{h}^1(\mathcal{I}_S(2)) = 1$. Then the cohomology table of S has the shape:

i

5				
		1		
		1	1	a
				$a+2$

p

$\mathrm{h}^i(\mathcal{I}_S(p))$

where $a \in \mathbb{N}$. Since the E^∞-terms outside the diagonal in Beilinson's spectral sequence vanish, there should exist a monomorphism

$$\Omega^2(2) \xrightarrow{\varphi} \mathcal{G} \oplus (a+2)\mathcal{O},$$

where $\mathcal{G}$ is the kernel of a morphism $\Omega^1(1) \xrightarrow{\psi} a\mathcal{O}$. Thus, by rank considerations, $a \geq 1$. On the other hand, if $a = 1$, then for such a monomorphism to exist, ψ must vanish identically. But then Beilinson's spectral sequence implies that Z, the codimension two subscheme defined as the degeneracy locus of φ, would be a component of S, which is a contradiction to the smoothness assumption. Therefore, necessary $a \geq 2$. We'll show that this leads to a contradiction. First, taking cohomology of the exact sequence

$$0 \longrightarrow \mathcal{I}_S(2) \longrightarrow \mathcal{I}_S(3) \longrightarrow \mathcal{I}_H(3) \longrightarrow 0$$

we see that $\mathrm{h}^1(\mathcal{I}_H(3)) = 2$, for all hyperplane sections H in a web. Since the general hyperplane section H in the web is smooth and $\mathcal{O}_H(3)$ non-special, we get $\mathrm{h}^0(\mathcal{O}_H(3)) = 21$, whence $\mathrm{h}^0(\mathcal{I}_H(3)) = 1$. Now, from the cohomology of the above exact sequence twisted by

H, we deduce that there are hyperplane sections H_0 in the web, for which $\mathrm{h}^1(\mathcal{I}_{H_0}(4)) = 2$. But $\mathrm{h}^1(\mathcal{O}_{H_0}(4)) = 0$, since $\mathrm{h}^1(\mathcal{O}_S(4)) = \mathrm{h}^2(\mathcal{O}_S(3)) = 0$, thus $\mathrm{h}^0(\mathcal{I}_{H_0}(4)) = 5$. This means that H_0 is contained in a cubic and an independent quartic surface, so it could be linked in their complete intersection to a curve of degree 1 and arithmetic genus -2, which is a contradiction.□

Therefore $\mathrm{h}^1(\mathcal{O}_S(2)) = \mathrm{h}^1(\mathcal{I}_S(2)) = 0$, and since a morphism $\Omega^1(1) \longrightarrow \mathcal{O}$ is never surjective (its cokernel has support on one point), we deduce that $\mathrm{h}^1(\mathcal{I}_S(4)) = 0$. Hence we obtain the cohomology table

i

5				
			1	
				2

p

$\mathrm{h}^i(\mathcal{I}_S(p))$

and thus S is the degeneracy locus of a morphism

$$0 \longrightarrow 5\mathcal{O}(-1) \longrightarrow \Omega^1(1) \oplus 2\mathcal{O} \longrightarrow \mathcal{I}_S(4) \longrightarrow 0.$$

In particular, S can be linked $(4,4)$ to an elliptic quintic scroll. Dualizing the last exact sequence we obtain

$$0 \longrightarrow \mathcal{O}(-5) \longrightarrow 2\mathcal{O}(-1) \oplus \Omega^3(3) \longrightarrow 5\mathcal{O} \longrightarrow \omega_S \longrightarrow 0,$$

so $|K|$ has no base points and S is minimal.□

Remark 3.61. *a) The homogenous ideal of an elliptic quintic scroll is generated by cubic hypersurfaces, so the scroll can be linked* $(4,4)$ *to a smooth surface as in the above proposition.*
b) The surfaces described in the previous proposition are hyperplane sections of smooth, log-general type, unirational threefolds $X \subset \mathbb{P}^5$, *with* $d = 11$, $\pi = 13$, $\chi = 1$ *and* $\kappa(X) = -\infty$ *(cf.* [BSS] *and* [Ch]*). An easy argument shows also the uniqueness of the examples given in* [BSS] *for threefolds with these invariants.*

Proposition 3.62. *There are no smooth surfaces* $S \subset \mathbb{P}^4$ *with invariants* $d = 11$, $\pi = 13$, $\chi = 7$.

Proof. We compute $\chi(\mathcal{O}_S(1)) = 6$, so by Riemann-Roch and Severi's theorem it follows that $\mathrm{h}^0(\mathcal{O}_S(K - H)) = \mathrm{h}^2(\mathcal{O}_S(H)) \geq 1$. But a curve $C \in |K - H|$ will have degree 2 and arithmetic genus 2 which is absurd.□

VII. Surfaces with $d = 11$, $\pi = 14$

In this last case we compute $K^2 = 6\chi - 32$ and $HK = 15$, so Hodge index implies $\chi \leq 8$. Lemma 0.20 yields $\chi \geq 1$ and, in case $\kappa(S) \geq 0$, the pseudo-effectiveness of K implies via (0.16) that $\chi \geq 3$ and in particular that $\kappa(S) \geq 1$.

Proposition 3.63. *There are no smooth rational surfaces with $d = 11$, $\pi = 14$.*

Proof. We use again the adjunction mapping to rule out the various candidates. If S_1 denotes the image of S under the adjunction mapping, we obtain the following invariants:

$S \subset \mathbb{P}^4$	$H^2 = 11$	$HK = 15$	$K^2 = -26$	$\pi = 14$
$S_1 \subset \mathbb{P}^{13}$	$H_1^2 = 15$	$H_1K_1 = -11$	$K_1^2 = -26 + a$	$\pi_1 = 3$,

where a is the number of the (-1)-lines on S. The adjoint linear system of S_1 has projective dimension two, so theorem 0.31 implies that either $(H_1 + K_1)^2 = a - 33 = 1$ and S_1 is $\mathbb{P}^2$ blown-up in one point, or $(H_1 + K_1)^2 = a - 33 = 0$ and S_1 is a conic bundle with one singular fibre in the ruling. In the first case we may write

$$H = 7l - 2E_0 - \sum_{i=1}^{34} E_i. \qquad (*)$$

In the second case, if $\mathbb{F}_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$, $e \geq 0$, is the relative minimal model of S_1, C_0 is a section of $\mathbb{F}_e$ with minimal self-intersection $-e$ and f is the class of a ruling, then working back through the adjunction process we may write

$$H \equiv 4C_0 + (2e+6)f - 2E_1 - \sum_{i=2}^{34} E_i.$$

Now $HC_0 \geq 1$ so $e \in \{0, 1, 2\}$, and we may choose $\mathbb{P}^2$ as minimal model

$$H = 8l - 4E_0 - 2E_1 - \sum_{i=2}^{34} E_i. \qquad (**)$$

We use cohomological information to exclude both cases. First, Riemann-Roch and Severi's theorem yield $h^1(\mathcal{O}_S(1)) = 6$, $\chi(\mathcal{I}_S(2)) = 7$ and $\chi(\mathcal{I}_S(4)) = 11$. Now the generic hyperplane section H of S is not hyperelliptic, so $h^1(\mathcal{O}_H(2)) = h^0(\mathcal{O}_H(K-H)) \leq 2$ by Clifford's theorem, and from the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_S(1) \longrightarrow \mathcal{O}_S(2) \longrightarrow \mathcal{O}_H(2) \longrightarrow 0$$

we obtain $h^1(\mathcal{O}_S(2)) \leq h^1(\mathcal{O}_S(1)) + 2 = 8$. Since $(K - 2H)H < 0$, we get also that $h^1(\mathcal{O}_H(k)) = 0$, for $k \geq 3$ and H an irreducible hyperplane section, and thus it follows from the cohomology of the above sequence that $h^1(\mathcal{O}_S(k)) \geq h^1(\mathcal{O}_S(k+1))$ for all $k \geq 2$. Therefore $h^1(\mathcal{O}_S(4)) \leq 8$, and since $\chi(\mathcal{I}_S(4)) = 11$ we deduce that S is contained in at least a net of irreducible quartic hypersurfaces. In particular the 5-secants of S can fill at most a surface of degree 5 in $\mathbb{P}^4$.

Assume now that S is embedded by the linear system $(*)$. Then residual to the curve E_0 there is a pencil $|D| = |H - E_0|$ of curves of degree 9 and sectional genus 12. If the general D is irreducible, then either it is a curve of type $(3, 5)$ on a smooth quadric, or it is linked $(2, 5)$ on a quadric cone with one of its rulings. In both cases the quadric would consist of 5-secants to S, and thus we obtain a contradiction since S would have too many 5-secant lines. If the general element of D is not irreducible, then the only possibility is that D has as base component a line L in this plane. Similar arguments rule also this case out.

Finally, assume that S would be embedded by the linear system $(**)$. Then a curve $C_i = l - E_0 - E_i$, for $i = \overline{2, 34}$, has degree 3 and thus spans only a hyperplane in $\mathbb{P}^4$. Residual to it there is a curve D_i of degree 8 and arithmetic genus 11. By (0.34), such a curve decomposes as $D_i = A_i + B_i$, where A_i is a plane sextic curve and B_i is a conic meeting A_i along a scheme of length 2. Therefore, by Bezout, the plane spanned by A_i lies in the intersection of all the quartic hypersurfaces containing S, and since the planes spanned by the various plane sextics are obviously distinct, the surface S would have again too many 5-secants.$\square$

Proposition 3.64. *There are no smooth surfaces $S \subset \mathbb{P}^4$ with invariants $d = 11$, $\pi = 14$ and $\chi = 3$.*

Proof. The claim is a consequence of the following refinement of the Castelnuovo inequality for space curves (cf. [Deb, Prop. 3.1])

Let X be a smooth non-ruled surface, and let L be a line bundle on X such that the induced mapping φ_L is birational on the image. Then

$$4\mathrm{h}^0(L) - 6 \leq \mathrm{h}^0(L^{\otimes 2})$$

which we apply for the adjoint linear system $|H + K|$ on S. Namely, in case such a surface would exist, we get $\mathrm{h}^0(\mathcal{O}_S(H + K)) = 16$, while $\mathrm{h}^0(\mathcal{O}_S(2H + 2K)) = \chi(\mathcal{O}_S(2H + 2K)) = 56$ since $\mathrm{h}^1(\mathcal{O}_S(2H + 2K)) = 0$ by Kodaira's vanishing theorem.$\square$

Proposition 3.65. *There are no smooth surfaces $S \subset \mathbb{P}^4$, with $d = 11$, $\pi = 14$ and $\chi = 4$.*

Proof. The double point formula yields $K^2 = -8$, while $p_g \geq 3$, thus S has at least 8 exceptional curves which are part of any canonical divisor on S. Let S_0, with canonical divisor K_0, denote the minimal model of S. Now S_0 is either elliptic or of general type.

If S_0 is elliptic, then K_0 is numerically equivalent to $nF + \sum_{i=1}^{k} a_i F_i$, $n > 1$ and $k, a_i \geq 0$, where F is a fibre of the elliptic fibration and the F_i's are the reduced parts of the multiple fibers. Since $HK_0 \leq 7$, we deduce that $n = 2$ and $HF = 3$. But then there is a curve $C \in |F - E|$, where E is one of the exceptional curves on S, with $HC \leq 2$ and $p_a(C) = 1$, which is a contradiction.

If S_0 is a surface of general type, then Noether's inequality gives

$$K_0^2 \geq 2p_g - 4 \geq 2,$$

thus S has at least ten (-1) curves E_i, $i = \overline{1,10}$, while $HK_0 \leq 5$ and $p_a(K_0) = K_0^2+1 \geq 3$. Since $p_g \geq 3$, we can find a curve $C \equiv K_0 - E_1 - E_2$ on S. It has degree $HC \leq 3$ and arithmetic genus $p_a(C) \geq 3$, and thus we have reached again a contradiction to the smoothness of S.□

Proposition 3.66. *There are no smooth surfaces $S \subset \mathbb{P}^4$, with $d = 11$, $\pi = 14$ and $\chi = 5$.*

Proof. We compute $K^2 = -2$, so S has at least two (-1) curves E_1 and E_2. As above, let S_0, with canonical divisor K_0, denote the minimal model of S.

If S_0 is elliptic, then we argue as in proposition 3.65 to obtain that $K_0 \equiv nF$, where F is a fiber of the elliptic fibration and $n \in \{2,3\}$. If $n = 3$, then $HF = 4$, $HE_1 = 1$ and $HE_2 = 2$, hence there exists a residual curve $D \in |F - E_2|$ of degree 2 and arithmetic genus $p_a(D) = 1$, which is impossible. If $n = 2$, then $HF = 4$, $HE_1 = 1$ and $HE_2 = 2$. The residual curve $C \in |F - E_2|$ has degree 4 and arithmetic genus 1, so it spans only a hyperplane in $\mathbb{P}^4$. Therefore there is a curve $D \in |H - C|$ of degree 7 and arithmetic genus 9. By lemma 0.33, D decomposes into a plane sextic curve A and a conic B, which doesn't meet A. The curve B is not exceptional, because otherwise it would be equal to E_2, while $B^2 = B^2 + AB = DB = DE_2 = (H - F + E_2)E_2 = 1$, which is a contradiction. Therefore $B^2 \leq -2$, and then $D^2 = A^2 + B^2 = 2$ implies $A^2 \geq 4$, which is impossible by the index theorem ($HA = 6$ implies that $A^2 \leq 3$).

We are left to consider the case when S_0 is a surface of general type. Then Noether's inequality yields $K_0^2 \geq 4$, so there are at least 6 exceptional curves E_i, $i = \overline{1,6}$, while $HK_0 \leq 9$ and $p_a(K_0) = K_0^2+1 \geq 5$. Since $p_g \geq 4$ we can find a curve $D \equiv K_0 - E_1 - E_2 - E_3$ on S. It has degree $HD \leq 6$ and arithmetic genus $p_a(D) = p_a(K_0) \geq 5$. Hence, either D is a plane quintic curve, $K_0^2 = 5$ and S has seven (-1) lines, or $HD = 6$, $p_a(D) = 5$ and S has six (-1) lines.
In the first case, residual to the plane quintic we get a pencil of curves of degree 6 and arithmetic genus 6. By (0.33), such a curve splits as the union of a plane quintic curve A and a line B, which meet in one point. Since the line doesn't move, it lies in the plane spanned by the quintic, so $B(A+B) = B(H-D) = 1 - BD = -4$, whence $B^2 = -5$. But $A^2 + B^2 = 1$, thus $A^2 \geq 6$, which contradicts the index theorem.
In the second case, D decomposes as the union of a plane quintic curve A and a line B, which does not meet A. As above $B^2 \leq -2$, $A^2 + B^2 = 1$, thus $A^2 \geq 3$ and we get again a contradiction by the index theorem.□

Proposition 3.67. *There are no smooth surfaces S of degree 11 in $\mathbb{P}^4$, with $\pi = 14$ and $\chi = 6$.*

Proof. The double point formula gives $K^2 = 4$, so S is of general type. On the other side $\mathrm{h}^2(\mathcal{O}_S(1)) = \mathrm{h}^0(\mathcal{O}_S(K - H)) = 0$ since $K^2 - H^2 < 0$, so Riemann-Roch and Severi's theorem yield for the speciality $h^1(\mathcal{O}_S(1)) = 1$. Now Clifford's theorem gives $h^1(\mathcal{O}_H(2)) = h^0(\mathcal{O}_H(K-H)) \leq 2$ for the generic hyperplane section H, whence $h^1(\mathcal{O}_S(2)) \leq 3$. On another side, Riemann-Roch and theorems 1.6 and 1.7 yield $h^1(\mathcal{O}_S(2)) \geq \chi(\mathcal{I}_S(2)) = 2$ and $h^1(\mathcal{O}_S(3)) \geq \chi(\mathcal{I}_S(3)) = 2$, while $h^1(\mathcal{O}_S(2)) \geq h^1(\mathcal{O}_S(3))$ since $(K-2H)H < 0$ and thus $h^1(\mathcal{O}_H(3)) = 0$ for an irreducible H. It follows that the variety $V \subset \check{\mathbb{P}}^4$ parametrizing hyperplane sections for which $h^1(\mathcal{O}_H(3)) > 0$ is ruled in lines or contains at least a plane. Now S is in any case a regular surface since otherwise, $K^2_{\text{min}} \geq 12$ by [Deb, Th.6.1] while $HK_{\text{min}} \leq 7$, $p_a(K_{\text{min}}) \geq 13$ and $p_g = 6$, which is absurd. Therefore an argument as in lemma 3.25. shows that there are only finitely many plane curves on S. In particular V cannot be ruled in lines since each line of the ruling would correspond to a $\mathbb{P}^2$ cutting S along a plane curve. On the other side, if V contains a plane π, then the line L, base locus of the net of hyperplanes parametrized by π, needs to be contained in S and thus we have an exact sequence

$$0 \longrightarrow \mathcal{O}_C(3H-L) \longrightarrow \mathcal{O}_H(3H) \longrightarrow \mathcal{O}_L(3H) \longrightarrow 0,$$

where $C = H - L$. The general element in $|C|$ is irreducible since otherwise $|C|$ would be composed with a basepoint free pencil of plane curves, thus $2p_a(C) - 2 \geq (H-L)(H-3L)$ which implies $L^2 \geq 3$, a contradiction.□

Proposition 3.68. *If S is a smooth surface of degree 11, with $\pi = 14$ and $\chi = 7$, then S is a minimal regular surface of general type, contained in a cubic hypersurface. S is linked to a Veronese surface in the complete intersection of the cubic and a quintic hypersurface.*

Proof. We have $K^2 = 10$, so S is of general type. Riemann-Roch gives $\chi(\mathcal{O}_S(1)) = 5$, thus Severi's theorem implies $h^1(\mathcal{O}_S(1)) = 0$, since $h^2(\mathcal{O}_S(1)) = h^0(\mathcal{O}_S(K-H)) = 0$ by lemma 0.17. The cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_S(1) \longrightarrow \mathcal{O}_S(2) \longrightarrow \mathcal{O}_H(2) \longrightarrow 0$$

yields then $h^1(\mathcal{O}_S(2)) = h^1(\mathcal{O}_H(2)) = h^0(\mathcal{O}_H(K-H))$. Since the general hyperplane section is not hyperelliptic, Clifford's theorem implies $h^1(\mathcal{O}_S(2)) \leq 2$. On the other side, Riemann-Roch and (1.6) give $h^1(\mathcal{O}_S(2)) \geq 1$.

Claim. $h^1(\mathcal{O}_S(3)) = 0$.

Proof of the claim. Assume the contrary. If $h^1(\mathcal{O}_S(2)) = 1$, then there would exist a web of hyperplane sections H such that $h^1(\mathcal{O}_H(3)) \neq 0$, which is absurd because the general element in the web is smooth and $\mathcal{O}_H(3)$ is non-special. If $h^1(\mathcal{O}_S(2)) = 2$, then S cannot be contained in a cubic hypersurface because theorem 1.7 implies that at most one of the two Hartshorne-Rao modules of S would be non-trivial in this case. Therefore Riemann-Roch gives $h^1(\mathcal{O}_S(3)) = 1 + h^1(\mathcal{I}_S(3))$. If $h^1(\mathcal{O}_S(3)) = 2$, then there exists a quadric cone such that, for all hyperplanes H containing one of its rulings, $h^1(\mathcal{O}_H(3)) = 1$. Each such line lies on S; otherwise the general hyperplane through it cuts out an integral curve and this is impossible ($(K-2H)H = -7 < 0$). But then it would follow that S is ruled, which is again impossible. If $h^1(\mathcal{O}_S(3)) = 1$, one uses Beilinson's spectral sequence to obtain a contradiction as in proposition 3.60. $\square$

Therefore $h^1(\mathcal{O}_S(3)) = 0$ and thus $h^0(\mathcal{I}_S(3)) \geq 1$. Theorem 1.7 implies then $h^1(\mathcal{O}_S(2)) = 1$, $h^1(\mathcal{O}_S(k)) = 0$, for $k \geq 3$, and $h^1(\mathcal{I}_S(k)) = 0$ for all k. In particular, it follows that S can be linked in the complete intersection of a cubic and a quintic hypersurface to a Veronese surface. All the other claims are immediate.$\square$

Remark 3.69. *Since the homogeneous ideal of the Veronese surfaces is generated by* 7 *cubics, we can link them* $(3,5)$ *to smooth surfaces as in the previous proposition.*

Proposition 3.70. *If S is a smooth surface of degree* 11, *with* $\pi = 14$ *and* $\chi = 8$, *then S is a minimal regular surface of general type. S can be linked in the complete intersection of two quartic hypersurfaces to a Castelnuovo surface of degree* 5, *thus it is projectively Cohen-Macaulay.*

Proof. Riemann-Roch and Severi's theorem give $h^2(\mathcal{O}_S(1)) = h^0(\mathcal{O}_S(K-H)) = 1 + h^1(\mathcal{O}_S(1)) > 0$, thus S is a minimal surface of general type. On the other side $H(K-H) = 4$ and $p_a(K-H) = 0$, so no part of $|K-H|$ can move on S, thus $h^2(\mathcal{O}_S(1)) = 1$ and $h^1(\mathcal{O}_S(1)) = 0$. Now the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_S(1) \longrightarrow \mathcal{O}_S(2) \longrightarrow \mathcal{O}_H(2) \longrightarrow 0$$

yields $h^1(\mathcal{O}_H(2)) = 1 + h^1(\mathcal{O}_S(2))$, while Clifford's theorem gives $h^0(\mathcal{O}_H(K-H)) \leq 2$. Hence $h^1(\mathcal{O}_S(2)) \leq 1$, and the argument we used in (3.56) shows that $h^1(\mathcal{O}_S(k)) = 0$, for all $k \geq 0$. In particular, $h^1(\mathcal{I}_S(3)) = h^0(\mathcal{I}_S(3)) = 0$ (by theorem 1.6) and $h^0(\mathcal{I}_S(4)) \geq 4$. If we cut to a general plane Π in $\mathbb{P}^4$, we get a linear system $|C|$ of plane quartic curves through 11 points, whose general member is irreducible. The projective dimension of $|C|$ is at least 4, and an argument similar to that used in [Ra1] for the proof of (0.34) shows that $|C|$ has only 11 base points. Thus the linear system of quartic hypersurfaces containing S has no base locus of codimension two outside S. By Bertini, we can link S in the complete intersection of two general quartic hypersurfaces to an irreducible surface $Y \subset \mathbb{P}^4$, with $d = 5$ and $\pi = 2$. The general hyperplane section H_Y of Y is an irreducible curve of degree 5 and arithmetic genus 2. Therefore H_Y is projectively normal, and so are Y and S. Hence Y is a Castelnuovo surface.$\square$

Remark 3.71. a) *The homogeneous ideal of Castelnuovo surfaces is generated by a quadric and two cubic hypersurfaces, hence they can be linked* $(4,4)$ *to smooth surfaces as described in the above proposition.*
b) *The above proposition yields an easy proof for the uniqueness of the examples provided in [BSS] of smooth threefolds, with invariants* $d = 11$ *and* $\pi = 14$. *In particular, we obtain that any such threefold has Kodaira dimension* 0 *and* $\chi(X) = 0$.

4. Examples of smooth surfaces of degree 12

We construct in the sequel several examples of smooth, non-general type surfaces of degree 12 in $\mathbb{P}^4$. In [DES] the authors constructed a smooth, blown-up $K3$ surface $S \subset \mathbb{P}^4$ with $d = 12$, $\pi = 14$, with one exceptional quartic and ten exceptional lines. We'll provide here examples of smooth, regular, proper elliptic surfaces.

Recall first that for a smooth surface S of degree 12 in $\mathbb{P}^4$ the double point formula reads

$$K^2 = 47 - 5\pi + 6\chi,$$

while Severi's theorem and Riemann-Roch give

$$\pi = \chi + 8 + \mathrm{h}^1(\mathcal{O}_S(H)) - \mathrm{h}^0(\mathcal{O}_S(K - H)).$$

Proposition 4.1. *There exist smooth, regular, minimal proper elliptic surfaces $S \subset \mathbb{P}^4$, with $d = 12$, $\pi = 13$, $\chi = 3$, and 10 skew 6-secant lines.*

Proof. For construction we use the Eagon-Northcott approach. A promising Beilinson cohomology table is

i ↑

2				
	2			
			4	5

→ p

$\mathrm{h}^i(\mathcal{I}_S(p))$

so we may set $\mathcal{F} = 2\mathcal{O}(-1) \oplus 2\Omega^3(3)$ and $\mathcal{G} = (\widetilde{\ker \psi})$, where $4R(1) \xleftarrow{\psi} 15R$ is the minimal free presentation of the cohomology module $\mathrm{H}^1(\mathcal{I}_S(* + 4)) = \mathrm{H}^1(\mathcal{G}(*))$, and try to construct the surface as the degeneracy locus of a general morphism $\varphi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$. However, for a general choice of the matrix ψ, the module $M = \operatorname{coker} \psi$ has a minimal free resolution of type

$$M \leftarrow 4R(1) \xleftarrow{\psi} 15R \leftarrow \begin{matrix} 15R(-1) \\ \oplus \\ 10R(-2) \end{matrix} \leftarrow 30R(-3) \leftarrow 21R(-4) \leftarrow 5R(-5) \leftarrow 0$$

and thus $\mathrm{Hom}(\Omega^3(3), \mathcal{G}) = 0$ in this case. What is needed is that the matrix defining ψ has at least two linear syzygies of second order. We'll choose ψ carefully.

Let F be the Horrocks-Mumford bundle on $\mathbb{P}^4$ (see [HM]). It is a stable rank 2 vector bundle with Chern classes $c_1 = -1$, $c_2 = 4$, and its H^1-cohomology module has a minimal free resolution of type (cf. [De1], [De2])

$$0 \leftarrow \mathrm{H}^1(F(*)) \leftarrow 5R \leftarrow 15R(-1) \nwarrow \begin{matrix} 10R(-2) \\ \oplus \\ 4R(-3) \\ \oplus \\ 15R(-4) \end{matrix} \nwarrow \begin{matrix} 2R(-3) \\ \oplus \\ 35R(-5) \end{matrix} \leftarrow 20R(-6) \nwarrow 2R(-8) \leftarrow 0.$$

We provide a somewhat different construction of this module in chapter 7. From [HM], or just by looking to the above resolution since $F(-1)$ is the cokernel of the unique morphism $0 \to 2\Omega^3(2) \xrightarrow{\theta} Syz_1(\mathrm{H}^1(F(*)))$, we have $\mathrm{h}^0(F) = \mathrm{h}^0(F(1)) = 0$ and $\mathrm{h}^0(F(2)) = 4$.

Consider now a rank 3 vector bundle E on $\mathbb{P}^4$ constructed as the extension

$$0 \longrightarrow F \longrightarrow E \longrightarrow \mathcal{O} \longrightarrow 0 \tag{4.2}$$

corresponding to a non-trivial element $0 \neq \xi \in \mathrm{H}^1(F) = \mathrm{Ext}^1(\mathcal{O}, F)$. E has Chern classes $c_1(E) = -1$, $c_2(E) = 4$ and $c_3(E) = 0$, and is stable because F is. Also $\mathrm{h}^1(E) = \mathrm{h}^1(F) - 1 = 4$ by construction. We'll assume in the sequel that the extension E comes from a generic element $\xi \in \mathrm{H}^1(F)$, meaning by this that ξ satisfies the following two conditions:

- the map $\mathbb{C}\xi \otimes \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(1)) \to \mathrm{H}^1(F(1))$ induced by multiplication with linear forms is injective, while
- the similar natural map $\mathbb{C}\xi \otimes \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(2)) \to \mathrm{H}^1(F(2))$ is surjective.

To see that such a choice is possible one either checks it for a random ξ with [Mac], or uses the invariance of $\mathrm{H}^1(F(*))$ under the group $G = \mathbb{H}_5 \rtimes \mathbb{Z}_2$, where $\mathbb{H}_5$ is the Heisenberg group of level 5. Namely, using notations and facts mentioned in (7.4), if $\mathbb{P}^4 = \mathbb{P}(V)$ then we can find a basis $e_0, \ldots, e_4$ of V such that, under the Schroedinger representation of $\mathbb{H}_5$ on $\mathbb{P}^4$, $\mathrm{H}^1(F) = V_3$, $\mathrm{H}^1(F(1)) = 2V_1^\sharp$, $\mathrm{H}^1(F(2)) = 2V_0^\sharp$, while the multiplication map $V^* \otimes \mathrm{H}^1(F) \to \mathrm{H}^1(F(1))$ is given by the projection on the second factor $V^* \otimes V_3 = 3V_1 \oplus 2V_1^\sharp \to 2V_1^\sharp$. One checks easily that $\xi = \sum_{i=0}^4 (-1)^i e_i \in \mathrm{H}^1(F)$ has the desired properties, and thus deduces that there is a Zariski open subset of $\mathrm{H}^1(F)$ fulfilling the two conditions.

With this general choice of ξ, the cohomology of the exact sequence (4.2) gives $\mathrm{h}^0(E(1)) = 0$, $\mathrm{h}^0(E(2)) = 5$, $\mathrm{h}^1(E(1)) = 5$ and $\mathrm{h}^1(E(m)) = 0$ for all $m \geq 2$ or $m \leq -1$. In conclusion, we deduce that $M = \mathrm{H}^1(E(*))$ is an artinian module with Hilbert function $(4, 5)$ and with the desired syzygies

$$M \leftarrow 4R \xleftarrow{\psi} 15R(-1) \nwarrow \begin{matrix} 15R(-2) \\ \oplus \\ 12R(-3) \end{matrix} \longleftarrow \begin{matrix} 2R(-3) \\ \oplus \\ 30R(-4) \end{matrix} \nwarrow 21R(-5) \leftarrow 5R(-6) \leftarrow 0$$

The two above linear syzygies of second order are inherited from those of $\mathrm{H}^1(F(*))$, and thus they involve too two proper Koszul complexes (see (7.5)).

As we can check in an example, the dependency locus of two general sections in $\mathrm{H}^0(E(2))$ is a smooth surface $S \subset \mathbb{P}^4$ with $d = 12$, $\pi = 13$ and $\chi = 3$:

$$0 \longrightarrow 2\mathcal{O} \longrightarrow E(2) \longrightarrow \mathcal{I}_S(5) \longrightarrow 0. \tag{4.3}$$

This description of S is equivalent to that as the degeneracy locus of a general morphism $\varphi \in \mathrm{Hom}(\mathcal{F}, Syz_1(\mathrm{H}^1(E(*+1))))$; we compute for S a minimal free resolution of type

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{array}{c} 3\mathcal{O}(-5) \\ \oplus \\ 12\mathcal{O}(-6) \end{array} \nwarrow 30\mathcal{O}(-7) \longleftarrow 21\mathcal{O}(-8) \longleftarrow 5\mathcal{O}(-9) \leftarrow 0.$$

Dualizing (4.3) we obtain

$$0 \longrightarrow \mathcal{O}(-5) \longrightarrow E^\vee(-2) \longrightarrow 2\mathcal{O} \longrightarrow \omega_S \longrightarrow 0$$

thus ω_S is globally generated, $p_g = 2$, and since the double point formula yields $K^2 = 0$ we deduce that S is a minimal proper elliptic surface. Le Barz's formula (0.11) gives $N_6 = 10$, hence there are ten 6-secant lines to S because $E(2)$ is globally generated outside a 3-codimensional set. $\square$

Corollary 4.4. *There exist smooth non-minimal $K3$ surfaces $X \subset \mathbb{P}^4$ with $d = 13$, $\pi = 16$ which are embedded by a linear system*

$$H = H_{min} - 7E_0 - \sum_{i=1}^{10} E_i.$$

Proof. The minimal proper elliptic surface $S \subset \mathbb{P}^4$ we've constructed above can be linked in the complete intersection of two quintic hypersurfaces to a surface X with invariants $d = 13$, $\pi = 16$, $\chi = 2$ and a resolution of type

$$0 \longrightarrow E^\vee(-2) \longrightarrow 4\mathcal{O} \longrightarrow \mathcal{I}_X(5) \longrightarrow 0. \tag{4.5}$$

Smoothness can be checked in an example. X is cut out by quintic hypersurfaces, hence there are no 6-secant lines. On the other hand, Le Barz's formula gives $N_6 = 10$, so there exist 10 exceptional lines on X, namely the 6-secant lines of S. Let now X_1 denote the image of X under the adjunction map, and X_2 denote the image of X_1 under the adjunction map defined by $|H_1 + K_1|$. From (0.13) we obtain the following invariants

$X_1 \subset \mathbb{P}^{16}$	$H_1^2 = 36$	$H_1K_1 = 6$	$K_1^2 = -1$	$\pi_1 = 22$
$X_2 \subset \mathbb{P}^{22}$	$H_2^2 = 47$	$H_2K_2 = 5$	$K_2^2 = -1 + b$	$\pi_2 = 27$,

where b is the number of (-1)-conics on X. Hodge index gives $K_2^2 = -1 + b \leq 0$, thus lemma 0.18 implies that X is either a $K3$ or a proper elliptic surface. Moreover, in case it is elliptic, X has a (-1) conic or a (-1) cubic and the proper transform of the canonical divisor on the minimal model is an elliptic curve of degree 4 or 5, while in case X is a $K3$ surface there is an exceptional rational septic curve on it. Now it is easily seen that in the case of an elliptic surface Kodaira's formula for the canonical divisor implies that $p_2 = \mathrm{h}^0({\omega_X}^{\otimes 2}) \geq 2$, so to prove the claim of the corollary we check that $p_2 = 1$. Dualizing (4.5) we obtain a resolution of ω_X

$$0 \longrightarrow \mathcal{O}(-5) \longrightarrow 4\mathcal{O} \longrightarrow E(2) \longrightarrow \omega_X \longrightarrow 0,$$

and thus one of the associated scandinavian complexes yields the resolution:

$$0 \longrightarrow 6\mathcal{O} \longrightarrow 4E(2) \longrightarrow S^2(E)(4) \longrightarrow {\omega_X}^{\otimes 2} \longrightarrow 0. \tag{4.6}$$

Splitting up (4.6) in short exact sequences and using the fact that $\mathrm{h}^1(E(2)) = 0$ and $\mathrm{h}^0(E(2)) = 5$ we obtain that $p_2 = \mathrm{h}^0(S^2(E)(4)) - 14$. Now, by (4.2), the bundle $S^2(E)$ can be realized as an extension

$$0 \longrightarrow S^2(F) \longrightarrow S^2(E) \longrightarrow E \longrightarrow 0$$

and since $\mathrm{h}^0(S^2(F)(4)) = 0$ and $\mathrm{h}^1(E(4)) = 0$ we need to compute the rank of the cobord morphism $\mathrm{H}^0(E(4)) \longrightarrow \mathrm{H}^1(S^2(F)(4))$. One checks in an example, via [Mac], that the kernel of the previous cobord is 15, thus $p_2 = 1$ and X is a $K3$-surface of the claimed type. From (4.5) we obtain also a minimal free resolution of type

$$0 \leftarrow \mathcal{I}_X \leftarrow \begin{array}{c} 4\mathcal{O}(-5) \\ \oplus \\ 5\mathcal{O}(-6) \end{array} \nwarrow 16\mathcal{O}(-7) \leftarrow 10\mathcal{O}(-8) \leftarrow 2\mathcal{O}(-9) \leftarrow 0$$

$\square$

Proposition 4.7. *There exist smooth, regular, proper elliptic surfaces $S \subset \mathbb{P}^4$, with invariants $d = 12$, $\pi = 14$, $\chi = 3$ and embedded by one of the following linear systems*

$$a) \qquad H = H_{min} - 2E_0 - \sum_{i=1}^{4} E_i,$$

$$b) \qquad H = H_{min} - \sum_{i=1}^{5} E_i.$$

Proof. A possible Beilinson cohomology table is

i

2				
	3	2		
			1	1

p

$\mathrm{h}^i(\mathcal{I}_S(p))$

Thus we may take $\mathcal{F} = 2\mathcal{O}(-1) \oplus 3\Omega^3(3)$ and $\mathcal{G} = \ker \psi$, for some epimorphism $\psi : 2\Omega^2(2) \oplus \Omega^1(1) \to \mathcal{O}$, and check for the degeneracy locus of a general morphism $\varphi \in \operatorname{Hom}(\mathcal{F}, \mathcal{G})$. Identifying $\mathbb{P}^4 = \mathbb{P}(V)$, with $V = \operatorname{span}_k(e_0, \ldots, e_4)$, the morphism ψ is induced by a triple $(\psi_{11}, \psi_{12}, \psi_2)$, where $\psi_{11}, \psi_{12} \in \Lambda^2 V$ and $\psi_2 \in V$. We check the various possibilities for ψ and see that only two of them lead to smooth surfaces. Namely, if we take

α) ψ generic, then the corresponding vector bundle $\mathcal{G}$ has a minimal free resolution of type

$$0 \leftarrow \mathcal{G} \leftarrow 25\mathcal{O}(-1) \nwarrow \begin{array}{c} 10\mathcal{O}(-2) \\ \oplus \\ 4\mathcal{O}(-3) \end{array} \longleftarrow \begin{array}{c} \mathcal{O}(-3) \\ \oplus \\ 4\mathcal{O}(-4) \end{array} \nwarrow \mathcal{O}(-5) \leftarrow 0,$$

and a general morphism $\varphi \in \operatorname{Hom}(\mathcal{F}, \mathcal{G})$ gives a smooth surface $S_\alpha \subset \mathbb{P}^4$ with minimal free resolution

$$0 \leftarrow \mathcal{I}_{S_\alpha} \leftarrow 8\mathcal{O}(-5) \nwarrow \begin{array}{c} 7\mathcal{O}(-6) \\ \oplus \\ 4\mathcal{O}(-7) \end{array} \longleftarrow \begin{array}{c} \mathcal{O}(-7) \\ \oplus \\ 4\mathcal{O}(-8) \end{array} \nwarrow \mathcal{O}(-9) \leftarrow 0$$

while, if we take

β) ψ distinguishing a plane, e.g., say $\psi_{11} = 0$, $\psi_{12} = e_0 \wedge e_1$ and $\psi_2 = e_2$, then we obtain a vector bundle $\mathcal{G}$ with resolution

$$0 \leftarrow \mathcal{G} \leftarrow \begin{array}{c} 25\mathcal{O}(-1) \\ \oplus \\ 2\mathcal{O}(-2) \end{array} \longleftarrow \begin{array}{c} 12\mathcal{O}(-2) \\ \oplus \\ 5\mathcal{O}(-3) \end{array} \longleftarrow \begin{array}{c} 2\mathcal{O}(-3) \\ \oplus \\ 4\mathcal{O}(-4) \end{array} \nwarrow \mathcal{O}(-5) \leftarrow 0,$$

and the generic $\varphi \in \operatorname{Hom}(\mathcal{F}, \mathcal{G})$ defines a smooth surface $S_\beta \subset \mathbb{P}^4$ with syzygies

$$0 \leftarrow \mathcal{I}_{S_\beta} \leftarrow \begin{array}{c} 8\mathcal{O}(-5) \\ \oplus \\ 2\mathcal{O}(-6) \end{array} \longleftarrow \begin{array}{c} 9\mathcal{O}(-6) \\ \oplus \\ 5\mathcal{O}(-7) \end{array} \longleftarrow \begin{array}{c} 2\mathcal{O}(-7) \\ \oplus \\ 4\mathcal{O}(-8) \end{array} \nwarrow \mathcal{O}(-9) \leftarrow 0.$$

Smoothness can be checked in examples on a computer via [Mac]. It is easily seen, as in the proof of (3.18), that all other choices of ψ lead to singular surfaces or to determinantal loci which are not in the expected codimension. We determine next what type of surfaces we've constructed.

Let now S_1 denote the image of S_α (or S_β resp.) under the adjunction map, and let S_2 be the image of S_1 under the adjunction map defined by $|H_1 + K_1|$. From (0.13) we obtain the following invariants

$$\begin{array}{lllll} S_1 \subset \mathbb{P}^{16} & H_1^2 = 35 & H_1K_1 = 9 & K_1^2 = -5 + a & \pi_1 = 23 \\ S_2 \subset \mathbb{P}^{24} & H_2^2 = 48 + a & H_2K_2 = 4 + a & K_2^2 = -5 + a + b & \pi_2 = 27 + a, \end{array}$$

where a is the number of (-1)-lines and b is the number of (-1)-conics on S_α (or S_β resp.).

In case α), the ideal I_{S_α} is generated by quintic hypersurfaces so S_α has no 6-secant lines and, since Le Barz's formula (0.11) gives $N_6 = 4$, there are 4 exceptional lines $E_1, E_2, \ldots E_4$

on S_α. Let $S_{\min}$ denote the minimal model of S_α and assume first that it is a surface of general type. Then S_α has at least two other exceptional curves F_1 and F_2 of degree ≥ 2, and thus there would exist a curve in $|K_{\min}-F_1|$ of degree $\leq HK_{\min}-2 \leq 4$ and arithmetic genus $p_a(K_{\min}) \geq 2$, which is a contradiction. It follows that S_α is a non-minimal elliptic surface, and thus $K \sim K_{\min} + \sum_{i=1}^{4} E_i + E_0$, where E_0 is a (-1) curve of degree ≥ 2. On the other side a curve in $|K_{\min} - E_0|$ has arithmetic genus one, so $HE_0 \leq 3$. We investigate first the case when $HE_0 = 3$. Then a curve $D \in |K_{\min} - E_0|$ has degree 4 and arithmetic genus one, so it spans only a hyperplane in $\mathbb{P}^4$. The residual curve $G \sim H - D$ has degree 8 and genus 9. We check now that G lies on a quadric surface in $\mathbb{P}^3$. Namely, since Riemann-Roch gives $\chi(\mathcal{O}_G(2H)) = 8$ it is enough to check that $\mathrm{h}^1(\mathcal{O}_G(2H)) \leq 1$. But this follows from the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_S(H+D) \longrightarrow \mathcal{O}_S(2H) \longrightarrow \mathcal{O}_G(2H) \longrightarrow 0$$

since, in the constructed example, $\mathrm{h}^1(\mathcal{O}_S(1)) = 3$, $\mathrm{h}^1(\mathcal{O}_S(2)) = 2$ and $\mathrm{h}^2(\mathcal{O}_S(H+D)) = \mathrm{h}^0(\mathcal{O}_S(K-D-H)) = 0$, while the composite multiplication map

$$\mathrm{H}^1(\mathcal{O}_S(H)) \xrightarrow{D} \mathrm{H}^1(\mathcal{O}_S(H+D)) \xrightarrow{H-D} \mathrm{H}^1(\mathcal{O}_S(2H))$$

drops rank at most one on $\check{\mathbb{P}}^4$, for a general choice of the morphism $\varphi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$. Now the curve D lies on two quadrics, thus for the hyperplane section H cut out by the $\mathbb{P}^3$ of D on S_α, we get $\mathrm{h}^0(\mathcal{I}_H(4)) \geq 2$. This is a contradiction, since under the previous assumption of minimal cohomology table we have $\mathrm{h}^0(\mathcal{I}_H(4)) \leq \mathrm{h}^1(\mathcal{I}_S(3)) = 1$, for all hyperplane sections H. Therefore E_0 is an exceptional conic and S_α is a non-minimal elliptic surface embedded by a linear system of type a), as claimed in the statement of the proposition.

In case β), it is easily seen that the distinguished plane $\Pi = \mathbb{P}(\mathrm{span}(e_0, e_1, e_2))$ meets S_β along a plane quintic curve C and the point $P = \mathbb{P}(\mathrm{span}(e_2))$ outside C. Therefore S_β has infinitely many 6-secant lines, namely all lines in Π going through P, and so Le Barz's formula doesn't apply in this case. In fact, using the explicit form of the syzygies of the ideal sheaf $\mathcal{I}_{S_\beta}$, it is easily seen that these are all the 6-secant lines to S_β. Taking cohomology of the exact sequences

$$0 \longrightarrow \mathcal{I}_{S_\beta}(k-1) \longrightarrow \mathcal{I}_{S_\beta}(k) \longrightarrow \mathcal{I}_H(k) \longrightarrow 0 \qquad k = \overline{3,4}$$

we observe that $\mathrm{h}^1(\mathcal{I}_H(3)) = 3$ for all hyperplane sections H of S_β, and that $\mathrm{h}^1(\mathcal{I}_H(4)) = 1$ if and only if $P \in H$. Therefore each hyperplane through P contains a plane π such that $\mathrm{h}^1(\mathcal{I}_{\pi \cap S_\beta}(4)) = 1$. In particular, for the pencil of hyperplanes through Π there exists a quadric cone

$$Q = \left\{ \det \begin{pmatrix} l & m \\ x_3 & x_4 \end{pmatrix} = 0 \right\},$$

where l and m are suitable linear forms, such that $\mathrm{h}^1(\mathcal{I}_{\pi_{(\lambda:\mu)} \cap S_\beta}(4)) = 1$ holds for all planes $\pi_{(\lambda:\mu)} = \{\mu l + \lambda m = \mu x_3 + \lambda x_4 = 0\}$ in one of the rulings of Q. Now the plane Π is obviously a member of the opposite ruling of Q, and thus residual to C in the complete intersection $S_\beta \cap Q$ there is a curve $G \sim 2H - C$ of degree 19 and arithmetic genus 23. On another side, residual to C, in the hyperplane sections through Π, there is a pencil $|D|$ with base point P, of curves of degree 7 and genus 3. It follows that the curve G splits as $G = G_1 + G_2$, where G_1 is a union of plane curves contained in planes of the ruling $\pi_{(\lambda:\mu)}$, while G_2 is a curve of degree 14 which maps down via projection from the vertex of Q to a complete intersection of type $(7,7)$ on the quadric surface which is the base of the cone. It is easily checked that G_1 splits as the union of 5 exceptional lines $E_1, E_2, \ldots, E_5$ on S_β. Therefore, on the first adjoint surface S_1 we obtain $K_1^2 \geq 0$, and in fact, by Hodge index, the equality $K_1^2 = 0$ holds. We argue further as in case α). If S_β would be a surface of general type then it contains further an exceptional curve E of degree ≥ 2 and thus a curve $N \in |K_{\min} - E|$ would have degree at most 5 and arithmetic genus at least 2. Thus the only possibility is $K_{\min}^2 = 1$, $HE = 2$ and $HN = 5$, $p_a(N) = 2$. If N spans only a $\mathbb{P}^3$, then the residual curve $H - N$ would have degree 7 and arithmetic genus 8, which is impossible by lemma 0.34. Therefore N spans all of $\mathbb{P}^4$ and necessarily splits as $N = A + B$, with A a plane quartic curve and B a line disjoint of it. But then $A^2 + B^2 = N^2 = 0$, $B^2 \leq -2$ since B cannot be exceptional, while Hodge index yields $A^2 \leq \frac{16}{12}$, and thus we obtained a contradiction. It follows that S_β is this time a non-minimal elliptic surface embedded by a linear system of type b).$\square$

5. Examples of Enriques surfaces of degree 13

We construct in this chapter two examples of smooth, non-minimal Enriques surfaces $S \subset \mathbb{P}^4$ with invariants $d = 13$, $\pi = 16$.

Lemma 5.1. *Let $S \subset \mathbb{P}^4$ be a smooth surface with $d = 13$, $\pi = 16$, $p_g = q = 0$. Then either:*
a) S is rational, or
b) S is a non-minimal Enriques surface with 17 exceptional lines.

Proof. Adjunction and the double point formula give $HK = 17$ and $K^2 = -17$. Let now $S_1 = \varphi_{H+K}(S) \subset \mathbb{P}^{15}$ be the image of S through the adjunction morphism. We obtain for its invariants

$$H_1^2 = 30, \qquad H_1K_1 = 0, \qquad \pi_1 = 16, \qquad K_1^2 = -17 + a$$

where a is the number of (-1) lines on S. Hence, since $H_1(nK_1) = 0$ for all n, it follows that either $p_2 = 0$ and S is rational by Castelnuovo's criterion, or $2K_1 \sim 0$ and S is an Enriques surface with 17 exceptional lines. $\square$

Remark 5.2. *Le Barz's 6-secant formula gives $N_6 = N_6(13, 16, 1) = 17$, thus a surface S which is cut out by quintic hypersurfaces is necessarily an Enriques surface of type b).*

(5.3.) Constructions. We provide in the sequel a construction of smooth Enriques surfaces as in the above lemma. A possible Beilinson cohomology table is

i

	6	5	1	
				1

p

$\mathrm{h}^i(\mathcal{I}_S(p))$

Riemann-Roch gives also $\mathrm{h}^2(\mathcal{O}_S(-1)) = \mathrm{h}^0(\mathcal{O}_S(H+K)) = \pi + \chi - 1 = 16$. Therefore, the above diagram suggests to construct $\mathcal{I}_S(4)$ as the cohomology of a quasi-monad

$$0 \longrightarrow 16\mathcal{O} \xrightarrow{\varphi} \mathcal{T} \xrightarrow{\psi} \mathcal{O} \longrightarrow 0$$

where $\mathcal{T}$ is an appropriate sheafified syzygy module of $M = \oplus_{m\in\mathbb{Z}} \mathrm{H}^2(\mathcal{I}_S(m+4))$. An artinian module M with a minimal free resolution of the form

$$\begin{array}{ccccccccccc}
0 \leftarrow M \longleftarrow 6R(3) \longleftarrow 25R(2) & & 36R(1) & & 16R & & & & \\
& \nwarrow_{\alpha} & \oplus & \longleftarrow & \oplus & & & & \\
& & R & & 10R(-1) & \nwarrow & 9R(-2) & & \\
& & & & & & & \nwarrow_{\beta} & R(-4) \leftarrow 0
\end{array}$$

would have the desired Hilbert function and would possibly fit our approach. To obtain a module with these syzygies it is easier to construct, as we did before, the k-dual module M^*. Namely, we look for nine quadrics without common zeroes, having a single second order linear syzygy. Moreover, the second order syzygy needs to properly involve 5 of the first order syzygies since we want the induced morphism $\mathcal{T}\xrightarrow{\psi}\mathcal{O}$ to be surjective. We can achieve this when, for instance, this part of the resolution comes from a truncated Koszul complex. Therefore we consider part of the Koszul complex

$$\begin{tikzcd} & & & R(-2) & \\ 0 & R \arrow[l] & 5R(-1) \arrow[l] & 10R(-2) \arrow[l] \arrow[u, "\delta"] & 10R(-3) \arrow[l, "\gamma"'] \arrow[ul, "\varepsilon"'] \end{tikzcd},$$

a general projection δ and the composition $\varepsilon = \delta \circ \gamma$. Then a general morphism in

$$\operatorname{Hom}(R(-4), \ker \varepsilon)$$

will give by composing with γ the nine quadrics defining β^*. Resolving β^* gives α. We take now $\mathcal{T} = Syz_2(M)$ and φ, ψ those morphisms defined by the resolution of M. In fact, if we denote the linear part of α by α' and put $\mathcal{E} = 16\mathcal{O}_{\mathbb{P}^4}$, $\mathcal{F} = \ker(36\mathcal{O}_{\mathbb{P}^4}(1)\xrightarrow{\alpha'}25\mathcal{O}_{\mathbb{P}^4}(2)) = \ker(\mathcal{T}\xrightarrow{\psi}\mathcal{O})$ and $\bar{\varphi} \in \operatorname{Hom}(\mathcal{E}, \mathcal{F})$ the morphism induced by $\varphi \in \operatorname{Hom}(16\mathcal{O}, \mathcal{T})$, then $\operatorname{coker} \bar{\varphi} = \mathcal{I}_S(4)$ is the twisted ideal sheaf of a smooth surface S with the desired invariants. We compute also a minimal free resolution of type

$$0 \leftarrow \mathcal{I}_S \longleftarrow \begin{matrix} 5\mathcal{O}(-5) \\ \oplus \\ \mathcal{O}(-6) \end{matrix} \longleftarrow 10\mathcal{O}(-7) \longleftarrow 6\mathcal{O}(-8) \longleftarrow \mathcal{O}(-9) \leftarrow 0$$

On the other hand, as we can check, S is cut out only by the quintics generating the homogeneous ideal I_S and therefore, by remark 5.2, S is a non-minimal Enriques surface with only 17 exceptional lines.

Remark 5.4 The above family was constructed in collaboration with Frank-Olaf Schreyer.

We describe in the sequel a variation of the above construction leading to a different family of Enriques surfaces. We start again with part of the Koszul complex

$$\begin{tikzcd} & & & 2R(-2) & \\ 0 & R \arrow[l] & 5R(-1) \arrow[l] & 10R(-2) \arrow[l] \arrow[u, "\delta"] & 10R(-3) \arrow[l, "\gamma"'] \arrow[ul, "\varepsilon"'] \end{tikzcd}$$

and, this time, with a general projection δ on $2R(-2)$ and consider the composition $\varepsilon = \delta \circ \gamma$. A general morphism in $\operatorname{Hom}(R(-4), \ker \varepsilon)$ will give by composing with γ eight quadrics cutting out a zero-dimensional scheme Z. We take as the nine quadrics defining β^* the previous eight plus a general one avoiding the scheme Z. The resulting module

M will have a minimal free resolution with the same Betti numbers. We put, as before, $\mathcal{T} = Syz_2(M)$ and consider the monad

$$0 \longrightarrow 16\mathcal{O} \xrightarrow{\varphi} \mathcal{T} \xrightarrow{\psi} \mathcal{O} \longrightarrow 0$$

where φ and ψ are those induced by the resolution of M. Its cohomology is the twisted ideal sheaf of a smooth surface S with the same invariants as above, having a minimal free resolution of type

$$0 \leftarrow \mathcal{I}_S \longleftarrow \begin{array}{c} 5\mathcal{O}(-5) \\ \oplus \\ 2\mathcal{O}(-6) \end{array} \longleftarrow \begin{array}{c} \mathcal{O}(-6) \\ \oplus \\ 10\mathcal{O}(-7) \end{array} \nwarrow 6\mathcal{O}(-8) \longleftarrow \mathcal{O}(-9) \leftarrow 0$$

We can check, as in the previous construction, that S is cut out only by the quintics generating the homogeneous ideal I_S and therefore, by remark 5.2, S is again a non-minimal Enriques surface with 17 exceptional lines. The extra linear syzygy of the quintics in I_S possess only 3 linearly independent components, which in turn define one of the exceptional lines of S.

Remarks 5.5.
a) An Enriques surface S belonging to one of the above two families is, by lemma 0.32*, the unique minimal scheme in its even liaison class.*
b) We were not able to find flat deformations from Enriques surfaces belonging to the first family to those of the second one, or to prove that the two constructed families reside in different components of the Hilbert scheme.
c) The existence, or non-existence of rational surfaces with these invariants is also completely open.

6. A $K3$ surface of degree 14

We construct in this chapter an example of a smooth non-minimal $K3$ surface of degree 14 in $\mathbb{P}^4$. Namely, we show

Proposition 6.1. *There exist smooth, non-minimal $K3$ surfaces $S = S_{min}(p_1, \ldots, p_{15}) \subset \mathbb{P}^4$ with $d = 14$, $\pi = 19$, $K^2 = -15$, and embedded via*

$$H = H_{min} - 4E_0 - \sum_{i=1}^{4} 2E_i - \sum_{j=5}^{14} E_j,$$

where $|H_{min}|$ is very ample on S_{min} of degree 56 and dimension 29.

Proof. We discuss first an approach as in chapter 1 via the Eagon-Northcott complex. A plausible Beilinson cohomology table for a surface with these invariants is

i ↑

1				
	7	7	3	

→ p

$\mathrm{h}^i(\mathcal{I}_S(p))$

thus, in this case, everything is determined by $M = \oplus_{m\in\mathbb{Z}} \mathrm{H}^2(\mathcal{I}_S(m+4))$. We'll look again to the dual module M^* and we'll assume that it is generated, as $R-$module, by $\mathrm{Hom}_k(\mathrm{H}^2(\mathcal{I}_S(3)), k)$. Then, by considering the Hilbert function of M^*, we find that M^* has a minimal free resolution of type

$$0 \longleftarrow M^* \longleftarrow 3R(-1) \xleftarrow{\psi} 8R(-2) \oplus mR(-3)$$

with $m \geq 0$. The morphism $\psi = (\psi_1, \psi_2)$ is given by a $3 \times (8+m)$ matrix with linear entries in ψ_1 and quadratic entries in ψ_2. Also $m > 0$ if and only if ψ_1 has at least three non-trivial linear syzygies. However, this doesn't occur for a general choice of ψ_1; in this case the cokernel of ψ_1 is an artinian graded module with Hilbert function $(3, 7, 5)$. In order to obtain a module with the desired Hilbert function, it is necessary that the number of linear syzygies of ψ_1 equals $m+2$, and we'll choose it correspondingly. The idea is to start with four planes $P_i = \{l_{i1} = l_{i2} = 0\}$, $i = \overline{1,4}$, and to consider the direct sum of the four Koszul complexes built on $\{\, l_{i1}, l_{i2} \,\}$

$$4R \xleftarrow{\alpha} 8R(-1) \xleftarrow{\beta} 4R(-2).$$

We take as ψ_1 a morphism given by 3 general lines of α, i.e., $\psi_1 = \gamma\alpha(-1)$, with $\gamma \in \mathrm{M}_{3,4}(k)$ a random matrix, and as ψ_2 a general 3×2 matrix with quadratic entries, because ψ_1 has exactly 4 linear syzygies. $M^* = \operatorname{coker} \psi$ is an artinian graded module with Hilbert function $(3, 7, 7)$ and from (1.11) we can compute its minimal free resolution

$$\begin{array}{ccccccc}
0 \leftarrow M^* \longleftarrow 3R(-1) & \underset{\psi}{\nwarrow} & 8R(-2) & & 4R(-3) & & \\
 & & \oplus & \nwarrow & \oplus & & \\
 & & 2R(-3) & & 5R(-4) & & \\
 & & & & \oplus & \nwarrow & \\
 & & & & 15R(-5) & & 38R(-6) \longleftarrow 28R(-7) \longleftarrow 7R(-8) \leftarrow 0
\end{array}$$

We dualize, and set according to proposition 1.15 $\mathcal{F} = Syz_2(M)$. To obtain the second bundle, we compare the syzygies of $\mathcal{F}$ with Beilinson's spectral sequence for $\mathcal{I}_S$. Namely, the E_∞-filtration yields an exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \oplus (\mathrm{H}^0(\mathcal{F}) \otimes \mathcal{O}) \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_S(4) \longrightarrow 0.$$

Furthermore, $\mathrm{h}^0(\mathcal{F}) = 15$, and if we put $\mathcal{E} = \mathcal{O}(-1) \oplus 15\mathcal{O}$, one checks, via [Mac] in examples, that the degeneration locus of a $\varphi \in \mathrm{Hom}(\mathcal{E}, \mathcal{F}) = \mathrm{Hom}(\mathcal{O}(-1) \oplus (\mathrm{H}^0(\mathcal{F}) \otimes \mathcal{O}), \mathcal{F}) = \mathrm{H}^0(\mathcal{F}(1)) \oplus \mathrm{Hom}(\mathrm{H}^0(\mathcal{F}) \otimes \mathcal{O}, \mathcal{F})$, given by a general section and the natural evaluation map, is a smooth surface S with the desired numerical invariants and the desired cohomology. The minimal free resolution of the ideal sheaf of the surface is of type

$$\begin{array}{cccccc}
 & 4\mathcal{O}(-5) & & 2\mathcal{O}(-6) & & \\
0 \leftarrow \mathcal{I}_S \longleftarrow & \oplus & \longleftarrow & \oplus & \nwarrow & \\
 & 4\mathcal{O}(-6) & & 8\mathcal{O}(-7) & & 3\mathcal{O}(-8) \leftarrow 0
\end{array}$$

so the homogeneous ideal is generated by 4 quintics and 4 sextics. Moreover, it follows from the construction of the module M^* that the four quintics containing S intersect in

$$V((I_S)_{\leq 5}) = S \cup \bigcup_{i=1}^{4} P_i,$$

and a closer look to the syzygies of M^* shows that S cuts each plane P_i along a sextic curve. Hence each of the planes P_i contains an ∞^2 of 6-secant lines, and in particular Le Barz's formula doesn't apply to this example.

To determine the type of surface we constructed, we check in an example that $S \cup \bigcup_{i=1}^4 P_i$ is an arithmetically Cohen-Macaulay scheme of degree 18 and sectional genus 39, with syzygies of type

$$\begin{array}{ccc}
0 \leftarrow \mathcal{I}_{S \cup \bigcup_{i=1}^4 P_i} \longleftarrow 4\mathcal{O}(-5) & & 2\mathcal{O}(-6) \\
 & \nwarrow & \oplus \quad \leftarrow 0 \\
 & & \mathcal{O}(-8)
\end{array}$$

and, moreover, that the minors of the 4×2 submatrix $4\mathcal{O}(-5) \xleftarrow{\alpha} 2\mathcal{O}(-6)$ vanish precisely along an exceptional quartic curve E_0 on S. Furthermore, by computing equations for the divisor contracted by the adjunction mapping, we see that there are ten exceptional lines on the surface. Let now S_1 denote the image of S under the adjunction map, and S_2 denote the image of S_1 under the map defined by $|H_1 + K_1|$. From (0.13) we obtain the following invariants

$S_1 \subset \mathbb{P}^{19}$	$H_1^2 = 43$	$H_1K_1 = 7$	$K_1^2 = -5$	$\pi_1 = 26$
$S_2 \subset \mathbb{P}^{26}$	$H_2^2 = 52$	$H_2K_2 = 2$	$K_2^2 = -5 + b$	$\pi_2 = 28,$

where b is the number of (-1)-lines on S_1. But $K_2^2 = -5 + b \geq -H_2K_2 = -2$, and there exists already an exceptional quartic curve E_0 on S, so the only possibility is that $b = 4$ and K_2 is a (-1) conic on S_2. As it turns out, $S = S_{min}(p_0, \ldots, p_{14})$ is a minimal $K3$ surface blown up in 15 points and

$$H = H_{min} - 4E_0 - \sum_{i=1}^{4} 2E_i - \sum_{j=5}^{14} E_j,$$

where H_{min} is a very ample linear system on $S_{min} = S_4$, giving an embedding $S_{min} \subset \mathbb{P}^{29}$ with $\deg S_{min} = 56$.

We want further to recover an alternative liaison construction for the above $K3$ surface. Proposition 0.30 and remark 0.31 ensure that we can link $(5, 5)$ the configuration $S \cup \bigcup_{i=1}^4 P_i$ to a smooth surface Y of degree 7 and sectional genus 6. The cohomology of the liaison exact sequence

$$0 \longrightarrow \mathcal{O}_Y(K_Y) \longrightarrow \mathcal{O}_{\Sigma_{5,5}}(5) \longrightarrow \mathcal{O}_{S \cup \bigcup_{i=1}^4 P_i}(5) \longrightarrow 0,$$

where $\Sigma_{5,5}$ denotes the complete intersection of the two quintic hypersurfaces used in the linkage, gives $p_g(Y) = 2$ and $q(Y) = 0$ while the double point formula yields $K_Y^2 = 0$. Surfaces with these invariants are classified in [Ok2] and are known to be arithmetically Cohen-Macaulay, minimal proper elliptic surfaces. Namely, $|K_Y|$ is a pencil without base points of plane cubic curves; the planes spanned by its members being those in one ruling of the determinantal quadric defined by the linear syzygies in

$$0 \longrightarrow 2\mathcal{O}(-5) \longrightarrow 2\mathcal{O}(-4) \oplus \mathcal{O}(-2) \longrightarrow \mathcal{I}_Y \longrightarrow 0.$$

Then liaison, once again, shows that Y cuts each plane P_i along a conic C_i, which is necessarily a section of the elliptic fibration, since there are no singular fibers and the fibration is by plane curves. In particular, this means that the rank of the Picard group of Y is at least 6, while the Picard number of a generic elliptic surface of degree 7 in $\mathbb{P}^4$ is only 2 by [ElP2]. Therefore we'll have to choose in the sequel Y carefully in order to recover S via liaison from the scheme $Y \cup \bigcup_{i=1}^4 P_i$.

(6.2.) Liaison construction. The above facts suggest us the following liaison method of construction for this family of $K3$ surfaces. Let P, P_1, P_2, P_3, P_4 be five planes in general position in $\mathbb{P}^4$ and denote by $\{p_{ij}\} = P_i \cap P_j$, for $1 \leq i < j \leq 4$, the mutual intersection points of the last four of them.

Lemma 6.3.
a) The homogeneous ideal $I_{P\cup\{p_{ij},1\le i<j\le 4\}}$ is generated by 3 quadrics and 4 cubics.
b) The three quadrics intersect along the plane P and a rational normal quartic curve Q, which is trisecant to P and goes through the points p_{ij}.

Proof. The first part follows from the cohomology of the residual intersection sequences

$$0 \longrightarrow \mathcal{I}_{\{p_{ij},i<j\}}(m-1) \longrightarrow \mathcal{I}_{P\cup\{p_{ij},i<j\}}(m) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(m-1) \longrightarrow 0$$

where $m \in \mathbb{Z}$. For the second part observe that the plane P is linked in the complete intersection of two of the quadrics to a rational cubic scroll T. If $H_T \sim C_0 + 2f$, with $C_0^2 = -1$, $C_0 f = 1$ and $f^2 = 0$, is the embedding of the scroll in $\mathbb{P}^4$, then $P \cap T \sim C_0 + f$ is a conic and thus the third quadric cuts on T the rational normal quartic curve $Q \sim C_0 + 3f$. Now $P \cap Q = Q(C_0 + f) = 3$ and the lemma follows.□

We consider now a general quadric $V \in \mathrm{H}^0(\mathcal{I}_{P\cup\{p_{ij}\}}(2))$ and denote with C_i the conics $V \cap P_i$, for $i = \overline{1,4}$. They intersect pairwise in the points $\{p_{ij}\} = C_i \cap C_j$, $1 \le i < j \le 4$.

Lemma 6.4. *There exists a unique rational normal quartic curve E_0 which is contained in V, passes through the points p_{ij}, $1 \le i < j \le 4$, and intersects the plane P in one point p.*

Proof. The claim is closely related to a theorem by James, see [Ja], [Sem]. For the proof we use an idea of Semple [Sem], [SR]. Consider the rational map $\gamma : \mathbb{P}^4 \dashrightarrow \mathbb{P}^5$ given by the quadrics through the rational normal quartic curve Q in (6.3). It is one to one onto a smooth hyperquadric Ω in $\mathbb{P}^5$, which we'll identify in the sequel with the image of the grassmannian of lines in $\mathbb{P}^3$ under the Plücker embedding. Let $\widetilde{\mathbb{P}}^4$ be the blowing up of $\mathbb{P}^4$ along Q and denote by E the exceptional divisor and by $\tilde{\gamma} : \widetilde{\mathbb{P}}^4 \longrightarrow \Omega \subset \mathbb{P}^5$ the induced morphism. Then the trisecant planes of Q are mapped through γ to the planes of one generating system, say α-planes, of the grassmannian Ω, while E is mapped by $\tilde{\gamma}$ onto a sextic threefold ruled in β-planes. Each of the β-planes corresponds to the normal directions in $\mathbb{P}^4$ at points of Q. We remark also that quadric cones through Q are mapped via γ to special linear complexes, i.e., to tangent hyperplane sections of Ω. To fix notations, let now $H \subset \mathbb{P}^5$ be the hyperplane corresponding to V and let β_{ij}, $1 \le i < j \le 4$, be the β-planes corresponding to the points p_{ij}. Rational normal quartic curves which meet Q in six points are represented via γ by conics in which Ω is met by planes. Thus, in order to prove the lemma, all we need to check is that in H exists exactly one plane meeting all six lines $H \cap \beta_{ij}$ and not contained in the quadric cone $H \cap \Omega$. But this is clear since, by Schubert calculus, the Plücker embedding of the grassmannian of planes in $\mathbb{P}^4$ has degree 5, while the planes of the cone $H \cap \Omega$ describe via the same Plücker embedding the union of two conics. The rational quartic curve E_0 represented by this unique plane meets P in one point because γ maps P in an α-plane contained in H.□

Lemma 6.5. *If $T = P \cup \bigcup_{i=1}^4 C_i$, then its homogeneous ideal I_T is generated by 1 quadric, 2 cubic and 4 quartic hypersurfaces.*

Proof. One uses again the residual exact sequences

$$0 \longrightarrow \mathcal{I}_{\cup_{i=1}^4 C_i}(m-1) \longrightarrow \mathcal{I}_T(m) \longrightarrow \mathcal{I}_{P\cup(\cup_{i=1}^4 C_i\cap H)}(m) \longrightarrow 0$$

where H is a general hyperplane through P and $m \in \mathbb{Z}$, together with the fact that $I_{\cup_{i=1}^4 C_i}$ is generated by 1 quadric and 8 cubic hypersurfaces.□

From the above lemma it follows that P can be linked in the complete intersection of V and a general quartic hypersurface $W \in \mathrm{H}^0(\mathcal{I}_T(4))$ to a smooth, minimal proper elliptic surface $Y \subset \mathbb{P}^4$ with $\deg Y = 7$, $\pi(Y) = 6$. Moreover, by construction, the conics C_i lie on Y.

Lemma 6.6.
a) On Y we have $C_i^2 = -3$ and $K_Y C_i = 1$, so each conic C_i is a section of the elliptic fibration.
b) The planes P_i intersect Y exactly along the conics C_i.

Proof. In any case $K_Y C_i \geq 1$ since there are no multiple fibers. On the other hand, we recall that the elliptic fibration is cut out on Y by the planes in one of the rulings of the cone V. Thus if $K_Y C_i \geq 2$, then P_i would lie on V and this would contradict our choices. It follows that $C_i^2 = -3$ and $K_Y C_i = 1$. Part *b*) is set theoretically clear by construction. The claim follows because residual to each conic C_i there is a pencil $|H_Y - C_i|$ of curves of degree 5 and genus 2, without base points since $(H_Y - C_i)^2 = 0$.□

We need in the sequel some classical facts of projective geometry.

Proposition (Segre) 6.7. *With any four general planes P_i, $i = \overline{1,4}$, there is associated a uniquely determined fifth plane P_5, such that all lines which meet the first four planes meet also the fifth.*

Proof. As mentioned above, the Plücker embedding of the grassmannian of lines in $\mathbb{P}^4$ has degree 5, thus the claim follows because the special linear complexes consist of lines meeting a given plane. See also [Seg] or [SR]. □

Corollary (Segre)[Seg] 6.8. *The lines in $\mathbb{P}^4$, which meet the four initial planes, generate a cubic hypersurface X containing the five planes P_i, $i = \overline{1,5}$, and having singularities (nodes) exactly at the ten points at which the planes meet in pairs.*

Proof. We briefly recall the arguments in [Seg]. The first part of the claim follows from Bezout since $\mathrm{H}^0(\mathcal{I}_{\cup P_i}(3)) = 1$ and from Schubert calculus in $G(1,4)$, since if l is a line meeting P_4 at one point q, in which case it is contained in a hyperplane H through P_4, then there is one line through q which meets P_1, P_2 and P_3, and there are two other lines meeting l and the three lines in which H cuts P_1, P_2 and P_3, respectively. To check singularities, observe first that residual to a plane P_i in a general hyperplane section of X through it, there is a quadric surface containing 4 skew lines, thus smooth. Therefore X has only isolated singularities and an easy argument shows that these are exactly the ten points of pairwise intersection of the planes P_i.□

A cubic threefold $X \subset \mathbb{P}^4$ with the maximum number of ordinary double points, namely 10, is unique up to projective equivalence (cf. [Seg], [Ka]). It's desingularization $\widetilde{X}$ is isomorphic to $\mathbb{P}^3$ blown-up in five points $a_1, \ldots, a_5$, in general position. The morphism

$\varphi : \widetilde{X} \to X \subset \mathbb{P}^4$ is given by the quadrics through the five points while the nodes are the images of the lines joining any two of the points a_i. We mention in the sequel some of the properties of this threefold (cf. [Seg], [SR], [Fi]).

The Segre cubic primal X has a symmetrical system of 15 planes, of which 5 correspond to the exceptional divisors over the points a_i, and 10 to the planes $P_{ijk} = \varphi(\mathrm{span}(a_i, a_j, a_k))$, for $\{i, j, k\} \subset \{1, 2, 3, 4, 5\}$. The symmetry of the the planes resides in the following properties:
- each plane contains four of the nodes,
- each plane is met in lines by 6 others, namely the plane corresponding to a_i by the planes P_{ijk}, for all $\{k, j\} \subset \{1, 2, 3, 4, 5\} \setminus \{i\}$, and the plane P_{ijk} by those corresponding to a_i, a_j, a_k and $P_{\alpha,\beta,\gamma}$, with $\alpha \in \{i, j, k\}$ and $\{\beta, \gamma\} = \{1, 2, 3, 4, 5\} \setminus \{i, j, k\}$.

We'll assume in the sequel that we've chosen the desingularization morphism φ such that the planes P_i, $i = \overline{1, 4}$, correspond to the exceptional divisors over a_i. Let now $Z = Y \cup \bigcup_{i=1}^4 P_i$. It is a local complete intersection scheme, except for the points P_{ij} which are Cohen-Macaulay of the type described in (0.31), and has invariants $\deg Z = 11$, $\pi(Z) = 10$, $\chi = 3$, $q = 0$. We remark here that Hodge index implies that there is no smooth surface in $\mathbb{P}^4$ with these invariants.
By computing syzygies one shows that Z has a resolution of type

$$0 \longrightarrow 2\mathcal{O}(-1) \oplus (\mathrm{H}^0(\mathcal{G}) \otimes \mathcal{O}) \longrightarrow \mathcal{G} \longrightarrow \mathcal{I}_Z(4) \longrightarrow 0$$

with $\mathcal{G} = Syz_1(M^*)(3)$, where M^* is the graded artinian module we constructed above. In particular the homogeneous ideal I_Z is generated by 3 quintic and 15 sextic hypersurfaces, and thus we can link Z in the complete intersection of two quintics to a surface S with $d = 14$, $\pi = 19$, $\chi = 2$, $q = 0$. One checks in examples via [Mac] that S is smooth.

Remark 6.9. *By liaison, each plane P_i, $i = \overline{1, 4}$, intersects S along a sextic curve D_i, thus each of them contains an ∞^2 of 6-secant lines.*

Lemma 6.10.
a) E_0 is an exceptional quartic on S.
b) The planes P_{ijk}, with $\{i, j, k\} \subset \{1, 2, 3, 4\}$, cut the surface S along four exceptional conics.

Proof. The rational normal curve E_0 is contained in V and intersects W in a scheme of length 16, of which one point is on P. Thus, for general choices, E_0 cuts $Z = Y \cup \bigcup_{i=1}^4 P_i$ along a scheme of length $15 + 6 = 21$ and, by Bezout, lies on all quintic hypersurfaces containing Z, whence on S. We show now that, say P_{123} cuts S along a conic; the other cases being similar. Observe first that P_{123} cuts P_1, P_2 and P_3 along the lines pairwise joining the points p_{12}, p_{13}, p_{23}, while P_4 and P_5 meet both this plane at the node v_{45} corresponding to the line through a_4 and a_5. For general choices P_{123} meets Y in a scheme of length 7: p_{12}, p_{13}, p_{23} and four extra points. Let E_4 denote the unique conic through these four points and the node v_{45}. It is easily seen that E_4 is a 11-secant conic to the configuration Z, so by Bezout it necessarily lies on S.

By linkage $K_S + ((Y \cup \bigcup_{i=1}^4 P_i) \cap S)_S \sim 5H_S$. On the other hand $(Y \cap S)_S \sim 5H_Y - K_Y - \sum_{i=1}^4 C_i$, which is a curve of degree 24 and arithmetic genus 37, and the quartic E_0 lie both on the quadric cone V. It follows that $K_S + ((\bigcup_{i=1}^4 P_i) \cap S)_S \sim 3H_S + E_0$, thus $K_S + \sum_{i=1}^4 D_i \sim (X \cap S)_S + E_0$. Since $H(K - E_0 - \sum_{i=1}^4 E_i) = 22 - 12 = 10$ and $K^2 = -15$ we deduce easily that E_i, $i = \overline{0,4}$, are exceptional curves on S.□

Adjunction and the above lemma show also that S must have 10 exceptional lines, thus it is a non-minimal $K3$ surface embedded by

$$H = H_{min} - 4E_0 - \sum_{i=1}^{4} 2E_i - \sum_{j=5}^{14} E_j.$$

Corollary 6.11. *The Segre cubic primal X intersects S along the union of* 10 *exceptional lines,* 4 *exceptional conics and* 4 *plane sextic curves.*

A similar liaison construction gives also the following

Proposition 6.12. *There exist smooth, non-minimal general type surfaces $S \subset \mathbb{P}^4$ with invariants $d = 15$, $\pi = 22$, $p_g = 3$, $q = 0$, $K^2 = -6$, and with* 9 *exceptional lines.*

Proof. We start this time with a Castelnuovo surface $Y \subset \mathbb{P}^4$, i.e., with a smooth, arithmetically Cohen-Macaulay, rational surface with $d = 5$, $\pi = 2$ and $K^2 = 1$ (see [Be] or [Ok1]). Y is linked to a plane in the complete intersection of a hyperquadric and a cubic hypersurface, and can be represented via the adjunction map as $\mathbb{F}_1$ blown up in 7 general points, thus it is embedded in $\mathbb{P}^4$ by

$$H_Y = 4l - 2E_0 - \sum_{i=1}^{7} E_i.$$

Consider now the following conics on Y:

$$\begin{aligned}
C_0 &= 3l - 2E_0 - \sum_{i=1}^{6} E_i \\
C_1 &= 2l - E_0 - E_7 - E_1 - E_3 - E_5 \\
C_2 &= 2l - E_0 - E_7 - E_1 - E_4 - E_6 \\
C_3 &= 2l - E_0 - E_7 - E_2 - E_4 - E_5 \\
C_4 &= 2l - E_0 - E_7 - E_2 - E_3 - E_6.
\end{aligned}$$

They intersect pairwise in one point and the planes they span, denoted in the sequel by P_i, for $i \in \{0, 4\}$, intersect the Castelnuovo surface Y only along the conics C_i. The scheme $Z = Y \cup \bigcup_{i=0}^4 P_i$ is regular, of degree 10 and sectional genus 7, and has a minimal free resolution of type

$$0 \leftarrow \mathcal{I}_Z \longleftarrow \begin{array}{c} 5\mathcal{O}(-5) \\ \oplus \\ 10\mathcal{O}(-6) \end{array} \nwarrow 34\mathcal{O}(-7) \longleftarrow 27\mathcal{O}(-8) \longleftarrow 7\mathcal{O}(-9) \leftarrow 0.$$

The five quintics in the ideal intersect along Z and the union of 9 skew lines. In fact, if, according to (6.7), Q_i denotes the unique Segre cubic hypersurface containing the planes P_{i+1}, P_{i+2}, P_{i+3} and P_{i+4}, for $i \in \mathbb{Z}_5$, then one checks that Q_1, Q_2, Q_3 and Q_4 each contain 6 skew lines which are 6-secant to the configuration Z, while Q_0 contains only 5 such lines. On the other hand, the five Segre cubics Q_i cut out an elliptic quintic scroll $T \subset \mathbb{P}^4$ (see [Seg], [SR, Th.XXXIII, p.278]); each of the planes P_i intersecting it along a cubic curve, section of the ruling. It follows that there are exactly 5 rulings of the scroll which are 6-secant to Z, and thus altogether 9 skew lines with this property. The scheme Z can be linked in the complete intersection of two quintic hypersurfaces to a surface S with the desired invariants, having the above 9 lines as exceptional curves. We compute the cohomology table

i ↑

3				
	7	8	4	

→ p

$\mathrm{h}^i(\mathcal{I}_S(p))$

and a minimal free resolution of type

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{matrix} 2\mathcal{O}(-5) \\ \oplus \\ 7\mathcal{O}(-6) \end{matrix} \nwarrow 12\mathcal{O}(-7) \longleftarrow 4\mathcal{O}(-8) \leftarrow 0.$$

Finally, we remark that each of the planes P_i intersects S along a sextic curve, and thus contains an ∞^2 of 6-secant lines to S.$\square$

Remark 6.13. *It follows from lemma* 0.32 *that S and Z are minimal elements in their even liaison classes.*

7. Construction of smooth surfaces with d=15, π=21, χ=0

First examples of surfaces with these invariants are due to Ellingsrud and Peskine; see [Au2]. Namely, for construction one starts with a minimal abelian surface $A \subset \mathbb{P}^4$ with $d = 10$, $\pi = 6$. Any such surface comes as the zero-set of a section of the (twisted) Horrocks-Mumford bundle $\mathcal{E}_{HM}(3)$, and thus it is Heisenberg invariant (with respect to $\mathbb{H}_5$, see below for notations) and lies on a net of Heisenberg invariant quintic hypersurfaces. The base locus of the net is made of 25 skew lines and one shows that A can be linked in the complete intersection of two such quintics to a smooth abelian surface S with $d = 15$, $\pi = 21$, having the 25 lines as (-1) curves, see [Au2]. A short discussion about a construction using the Eagon-Northcott complex method can be found in [DES].

In the sequel we recall the determinantal construction of this example from a different point of view and an upshot of our approach will be the construction of a new abelian surface lying on only one quintic hypersurface. A rather striking fact, which was the starting point of this investigation, is the existence of a degeneration of this abelian surface which is scheme-theoretically the first infinitesimal neighborhood of an elliptic quintic scroll. This chapter owes very much to the discussions I had with the authors of [ADHPR] during the preparation of that paper.

Concerning the possible invariants of such surfaces we mention the following

Lemma 7.1. *Let $S \subset \mathbb{P}^4$ be a smooth surface with $d = 15$, $\pi = 21$, $\chi = 0$. Then either:*
a) S is a ruled surface over an elliptic curve, or
b) S is a non-minimal bielliptic surface embedded by

$$H = H_{min} - \sum_{i=1}^{25} E_i,$$

or
c) S is a non-minimal abelian surface embedded by

$$H = H_{min} - \sum_{i=1}^{25} E_i.$$

Proof. Adjunction and the double point formula give $HK = 25$ and $K^2 = -25$. Therefore, when $p_g \geq 1$, it follows that S has exactly 25 exceptional lines and the canonical class on the minimal model needs to be trivial, whence *b*). Assume now $p_g = 0$ and let $S_1 = \varphi_{H+K}(S) \subset \mathbb{P}^{19}$ be the image of S through the adjunction morphism. The invariants of S are

$$H_1^2 = 40, \qquad H_1K_1 = 0, \qquad \pi_1 = 21, \qquad K_1^2 = -25 + a$$

where a is the number of (-1) lines on S. In particular $H_1(nK_1) = 0$ for all n, so if $\kappa(S) \geq 0$ then some multiple of K_1 is trivial, $a = 25$ and S is a bielliptic surface as claimed in *c*), otherwise S needs to be ruled over an elliptic curve.□

Remark 7.2.
a) I know nothing about the existence of ruled surfaces with the above invariants.
b) Examples of smooth non-minimal bielliptic surfaces as in lemma 7.1, b) have been constructed in [ADHPR].
b) Le Barz's 6-secant formula gives $N_6 = N_6(15, 21, 0) = 25$ so an abelian or a bielliptic surface as in the lemma has either none or infinitely many 6-secant lines.

7.3. We recall briefly some classical facts about the Heisenberg group $\mathbb{H}_5$. Let, as usual, $\mathbb{P}^4 = \mathbb{P}(V)$, with $V = <e_0, e_1, \ldots, e_4>$ and define $\mathbb{H}_5 \subset \mathrm{SL}(V)$ as the subgroup generated by σ and τ, where

$$\sigma(e_i) = e_{i-1}, \qquad \tau(e_i) = \xi^i e_i$$

for all $i \in \mathbb{Z}_5$ and $\xi = e^{\frac{2\pi i}{5}}$. In fact $[\sigma, \tau] = \xi$, so $\mathbb{H}_5$ is a central extension

$$0 \longrightarrow \mathbb{Z}_5 \longrightarrow \mathbb{H}_5 \longrightarrow \mathbb{Z}_5 \times \mathbb{Z}_5 \longrightarrow 0.$$

The representation $\mathbb{H}_5 \subset \mathrm{GL}(V)$ described above is called the Schrödinger representation and will be denoted in the sequel as V_0. Let θ, with $\theta(\xi) = \xi^2$, be the generator of $\mathrm{Gal}(\mathbb{Q}(\xi) : \mathbb{Q})$ and denote by V_i the composition $\mathbb{H}_5 \xrightarrow{\theta^i} \mathbb{H}_5 \xrightarrow{V_0} \mathrm{GL}(V)$. Then V_0, V_1, V_2, V_3 together with the 25 characters of $\mathbb{Z}_5 \times \mathbb{Z}_5$ form a basis of irreducible representations of $\mathbb{H}_5$ [HM]. Let now $\iota \in N_{\mathbb{H}_5|\mathrm{SL}_5(\mathbb{C})}$ be the standard Heisenberg involution $\iota(e_i) = e_{-i}$ and let $G = \mathbb{H}_5 \rtimes \mathbb{Z}_2$ be the subgroup generated by $\mathbb{H}_5$ and ι in the normalizer $N_{\mathbb{H}_5|\mathrm{SL}_5(\mathbb{C})}$. Let also V^+, resp. V^- be the eigenspaces corresponding to the eigenvalues 1, resp. -1 of the Heisenberg involution ι. As usual, we denote $\mathbb{P}^2_+ = \mathbb{P}(V^+)$ and $\mathbb{P}^1_- = \mathbb{P}(V^-)$. We will make use in the sequel of the representation theory of G. For convenience, we list here its character table [HM], [Ma]:

Z_α	$C_{m,n}$	C_p	
1	1	1	I
$5\theta^i(\alpha)$	0	$\theta^i(\xi^p)$	V_i
1	1	-1	S
$5\theta^i(\alpha)$	0	$-\theta^i(\xi^p)$	$V_i^\sharp$
2	$\xi^{sn+tm} + \xi^{-sn-tm}$	0	$Z_{s,t}$

where the conjugacy classes of G are

$$Z_\alpha = \{\ \alpha\ \}, \quad \text{with } \ \alpha \in Z(\mathbb{H}_5) = \mathbb{Z}_5 = \{\ 1, \xi, \ldots, \xi^4\ \} \qquad \text{(5 items)}$$

$$C_{m,n} = \{\ \xi^{2mn+p}\sigma^m\tau^n\ ,\ \xi^{2mn+p}\sigma^{-m}\tau^{-n}\ \mid\ p \in \mathbb{Z}_5\ \} \qquad \text{(12 items)}$$

and

$$C_p = \{\ \xi^{2mn+p}\sigma^m\tau^n\iota\ \mid\ m, n \in \mathbb{Z}_5\ \} \qquad \text{(5 items).}$$

There are 12 different irreducible representations $Z_{s,t}$ and we denote by Z their direct sum. We recall also the following formulae from [Ma]:

$$V_i \otimes V_i = 3V_{i+1} \oplus 2V^\sharp_{i+1}, \qquad V_i \otimes V_{i+1} = 3V_{i+3} \oplus 2V^\sharp_{i+3}, \qquad V_i \otimes V_{i+2} = I \oplus Z$$

$$(V_i)^* = V_{i+2}, \qquad (V_i^\sharp)^* = V_{i+2}^\sharp, \qquad V_i \otimes S = V_i^\sharp, \qquad V_i \otimes Z = 12V_i \oplus 12V_i^\sharp$$

where $i \in \mathbb{Z}_4$.

For the exterior powers the following isomorphisms hold

$$\overset{2}{\Lambda} V_0 = 2V_1^\sharp, \qquad \overset{3}{\Lambda} V_0 = 2V_3^\sharp, \qquad \overset{4}{\Lambda} V_0 = V_2$$

hence the Koszul complex, as complex of G modules, looks

$$0 \leftarrow k \leftarrow R \leftarrow V_2 \otimes R(-1) \leftarrow 2V_3^\sharp \otimes R(-2) \leftarrow 2V_1^\sharp \otimes R(-3) \leftarrow V_0 \otimes R(-4) \leftarrow R(-5) \leftarrow 0.$$

Finally, the well-known formula for the characters of the symmetric powers

$$\chi_{S^m V} = \sum_{i=1}^{5} (-1)^{i+1} \chi_{S^{m-i} V \otimes \Lambda^i V}$$

yields

$$\mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) = V_2, \qquad \mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) = 3V_3, \qquad \mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)) = 5V_1 \oplus 2V_1^\sharp$$

$$\mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(4)) = 10V_0 \oplus 4V_0^\sharp, \qquad \mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5)) = 6I \oplus 5Z.$$

7.4. The H^1-module of the Horrocks-Mumford bundle $\mathcal{E}_{HM}$ is a well understood module with Hilbert function $(5, 10, 10, 2)$. We construct in the sequel other Heisenberg invariant, graded artinian modules M with Hilbert function of type $(5, 10, 10, a)$, with $a \geq 0$, and we make use for that of the double complex described in chapter 1.

Namely, assume that M is normalized such that $M_n = 0$ for $n < 0$ and that $M_0 \cong V_3$ as G-representation spaces. Assume also that M_0 generates M as R-module. The previous formulae yields that, in terms of G-modules, the top part of the double complex in theorem 1.11 looks like

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
V_3 \otimes_k R & (3V_1 \oplus 2V_1^\sharp) \otimes_k R(-1) \arrow[l, "\mathrm{1I}_{M_0} \otimes \mathrm{d}_1"'] \arrow[d, "(\delta_1)_0 \otimes \mathrm{1I}_{R(-1)}"] & (4V_0 \oplus 6V_0^\sharp) \otimes_k R(-2) \arrow[l] \arrow[d, "(\delta_2)_0 \otimes \mathrm{1I}_{R(-2)}"] & (2S \oplus 2Z) \otimes_k .. \arrow[l] \arrow[d] \\
0 \arrow[u, leftarrow] \arrow[d] & M_1 \otimes_k R(-1) \arrow[l] \arrow[d] & M_1 \otimes_k V_2 \otimes_k R(-2) \arrow[l] \arrow[d, "(\delta_1)_1 \otimes \mathrm{1I}_{R(-2)}"] & M_1 \otimes_k 2V_3 \otimes_k .. \arrow[l] \arrow[d] \\
0 \arrow[d] & 0 \arrow[l] \arrow[d] & M_2 \otimes_k R(-2) \arrow[l] \arrow[d] & M_2 \otimes_k V_2 \otimes_k .. \arrow[l] \arrow[d] \\
{} & {} & {} & {}
\end{tikzcd}$$

The compatibility of the multiplication map $(\delta_1)_0 \otimes \mathrm{1I}_{\mathrm{R}(-1)}$ with the action of G on M implies that we have roughly speaking exactly three choices, as G-module, for the 10-dimensional vector space M_1:

a) $M_1 = 2V_1$, or

b) $M_1 = 2V_1^\sharp$, or

c) $M_1 = V_1 \oplus V_1^\sharp$,

We'll discuss each case separately. The presentation matrix σ_1 of M which comes out from the homology spectral sequence of the double complex can be easily recovered in all cases. Namely, fix first a splitting $M_0 \otimes_k V_2 = \oplus_{i=0}^4 L_i$ with $L_i \cong V_1$ for $i = \overline{0,2}$ and $L_j \cong V_1^\sharp$ for $j = \overline{3,4}$. The general $\mathbb{H}_5$ invariant subspace of $M_0 \otimes_k V_2$ can be displayed as $L^a = a_0 L_0 + a_1 L_1 + \cdots + a_4 L_4$ for some $a = (a_0 : a_1 : \ldots : a_4) \in \mathbb{P}^4$, where the above sum means component-wise addition of the bases with the prescribed a_j as coefficients. The restriction of $1\!\!1_{\mathrm{M}_0} \otimes \mathrm{d}_1$ to $L^a \otimes_k R(-1)$ is given by the 5×5 matrix with linear entries

$$\Phi_a = \begin{pmatrix} x_0a_0 & x_3(a_1-a_4) & x_1(a_2-a_3) & x_4(a_2+a_3) & x_2(a_1+a_4) \\ x_3(a_1+a_4) & x_1a_0 & x_4(a_1-a_4) & x_2(a_2-a_3) & x_0(a_2+a_3) \\ x_1(a_2+a_3) & x_4(a_1+a_4) & x_2a_0 & x_0(a_1-a_4) & x_3(a_2-a_3) \\ x_4(a_2-a_3) & x_2(a_2+a_3) & x_0(a_1+a_4) & x_3a_0 & x_1(a_1-a_4) \\ x_2(a_1-a_4) & x_0(a_2-a_3) & x_3(a_2+a_3) & x_1(a_1+a_4) & x_4a_0 \end{pmatrix}$$

Therefore, introducing R.Moore's matrices $M_y(x) = (x_{3i+3j}y_{i-j})_{i,j}$, with $i,j \in \mathbb{Z}_5$, Φ_a can be expressed as

$$\Phi_a = M_z(x)$$

where $z \in \mathbb{P}^4$ has the following components:

$$z_0 = a_0, \quad z_1 = a_1 + a_4, \quad z_2 = a_2 + a_3, \quad z_3 = a_2 - a_3, \quad z_4 = a_1 - a_4.$$

It follows that the restriction of $1\!\!1_{\mathrm{M}_0} \otimes \mathrm{d}_1$ to a L^a of type $V_1^\sharp$ corresponds to a matrix $\Phi_a = M_z(x)$ with $z \in \mathbb{P}^2_+$, while the restriction to an L^a of type $V_1^\sharp$ corresponds to a matrix $\Phi_a = M_z(x)$ with $z \in \mathbb{P}^1_-$. To summarize, in case *a)*

$$\sigma_1 = \big(M_z(x) \mid M_{z_1}(x) \mid M_{z_2}(x)\big)$$

with $z \in \mathbb{P}^2_+$, $z_1 = (0,1,0,0,-1) \in \mathbb{P}^1_-$, $z_2 = (0,0,1,-1,0) \in \mathbb{P}^1_-$, while in case *b)*

$$\sigma_1 = \big(M_{z_0}(x) \mid M_{z_1}(x) \mid M_{z_2}(x)\big)$$

with $z_i \in \mathbb{P}^2_+$, $i = \overline{0,2}$ and thus, without loss of generality, we can choose say $z_0 = (1,0,0,0,0)$, $z_1 = (0,1,0,0,1)$ and $z_2 = (0,0,1,1,0)$, and in case *c)*

$$\sigma_1 = \big(M_{z_1}(x) \mid M_{z_2}(x) \mid M_z(x)\big)$$

with $z_i \in \mathbb{P}^2_+$, $i = \overline{1,2}$ and $z \in \mathbb{P}^1_-$.

We look first to the case *b)*. The vertical differential $(\delta_3)_0 \otimes 1\!\!1_{\mathrm{R}(-1)} : (2\mathrm{S} \oplus 2\mathrm{Z}) \longrightarrow \mathrm{M}_1 \otimes_\mathrm{k} 2\mathrm{V}_3 = 4\mathrm{I} \oplus 4\mathrm{Z}$ has a two-dimensional kernel. Since the two copies of S involve two Koszul complexes in the top row of the double complex it follows that the differential $(\delta_3)_0 \otimes \mathbb{1}_{\mathrm{R}(-1)} : 4\mathrm{V}_0 \oplus 6\mathrm{V}_0^\sharp \longrightarrow \mathrm{M}_0 \otimes_{\mathrm{k}} \mathrm{V}_2 = 4\mathrm{V}_0 \oplus 6\mathrm{V}_0^\sharp$ has a 10-dimensional kernel of type $2V_0^\sharp$. One checks easily that $M_2 \cong 2V_0^\sharp$ and $M_3 \cong 2S$, while $M_k = 0$ for $k \geq 4$. Therefore M has Hilbert function $(5, 10, 10, 2)$ and a minimal free resolution of type

$$\begin{array}{ccccccccc}
M \longleftarrow 5R \xleftarrow{\sigma_1} 15R(-1) & \nwarrow & 10R(-2) & & & & & & \\
& & \oplus & & & & & & \\
& & 4R(-3) & \nwarrow & 2R(-3) & & & & \\
& & \oplus & & \oplus & & & & \\
& & 15R(-4) & & 35R(-5) & \longleftarrow & 20R(-6) & \nwarrow & \\
& & & & & & & & 2R(-8) \longleftarrow 0
\end{array}$$

This module is precisely (up to a twist) the H^1-module of the Horrocks-Mumford bundle. See also [De2] for a presentation of this module and [DES] for the Eagon-Northcott construction using the syzygy bundles of this module of the abelian surface in degree 10 and its $(5, 5)$ linked abelian surface of degree 15.

In case *c)*, for a generic projection on a subspace of type $V_1 \oplus V_1^\sharp$, the vertical differential $(\delta_2)_0 \otimes \mathbb{1}_{\mathrm{R}(-1)} : 4\mathrm{V}_0 \oplus 6\mathrm{V}_0^\sharp \longrightarrow \mathrm{M}_0 \otimes_{\mathrm{k}} \mathrm{V}_2 = 5\mathrm{V}_0 \oplus 5\mathrm{V}_0^\sharp$ has only a 5-dimensional kernel. It is easy to check that the general such module will have Hilbert function $(5, 10, 5)$. However, special projections give modules with Hilbert function $(5, 10, 10, 1)$, or $(5, 10, 10)$, namely the first type being the H^1-module of the ideal sheaf of a union of two elliptic quintic scrolls meeting along an elliptic normal curve in $\mathbb{P}^4$ (construction due to Frank-Olaf Schreyer). Since we'll make no use of this we don't insist on further details.

In case *a)*, for a general projection on $2V_1$, the differential $(\delta_2)_0 \otimes \mathbb{1}_{\mathrm{R}(-1)} : 4\mathrm{V}_0 \oplus 6\mathrm{V}_0^\sharp \longrightarrow \mathrm{M}_0 \otimes_{\mathrm{k}} \mathrm{V}_2 = 6\mathrm{V}_0 \oplus 4\mathrm{V}_0^\sharp$ has a 10-dimensional kernel isomorphic with $2V_0^\sharp$. Its cokernel is M_2, whence $M_2 \cong 2V_0$. A similar argument shows that $M_3 \cong 2I$ and $M_k = 0$ for $k \geq 4$. Therefore, M has in this case again Hilbert function $(5, 10, 10, 2)$, but this time a different minimal free resolution. Namely, one gets

$$\begin{array}{ccccccccc}
M \longleftarrow 5R \xleftarrow{\sigma_1} 15R(-1) & \nwarrow & 10R(-2) & & & & & & \\
& & \oplus & & & & & & \\
& & 2R(-3) & & & & & & \\
& & \oplus & \nwarrow & & & & & \\
& & 15R(-4) & & 35R(-5) & \longleftarrow & 20R(-6) & \nwarrow & \\
& & & & & & & & 2R(-8) \longleftarrow 0
\end{array}$$

We use the syzygy bundles of the dual module to construct a smooth, non-minimal abelian surface of degree 15 which is not coming via a $(5, 5)$ linkage from the Horrocks-Mumford torus. A plausible cohomology table for an abelian surface S of degree 15, sectional genus 21 in $\mathbb{P}^4$ is the following:

i

1				
2	10	10	5	

p

$\mathrm{h}^i(\mathcal{I}_S(p))$

Let now $\mathcal{E} = \mathcal{O}_{\mathbb{P}^4}(-1) \oplus (2V_2 \oplus V_2^\sharp) \otimes_k \mathcal{O}_{\mathbb{P}^4} = \mathcal{O}_{\mathbb{P}^4}(-1) \oplus 15\mathcal{O}_{\mathbb{P}^4}$ and $\mathcal{F} = S \otimes_k Syz_2(M^*)(1)$. One checks in examples that the degeneracy locus of a general morphism $\varphi \in \mathrm{Hom}(\mathcal{E}, \mathcal{F})$ is a smooth surface $S \subset \mathbb{P}^4$ with $d = 15$, $\pi = 21$, $p_g = 1$, $q = 2$, $K^2 = -25$ which, by lemma 7.1, is thus a non-minimal abelian surface with 25 exceptional lines. The minimal free resolution of the ideal sheaf is of the form

$$0 \longleftarrow \mathcal{I}_S \longleftarrow \begin{array}{c} I \otimes_k \mathcal{O}(-5) \\ \oplus \\ 2V_2 \otimes_k \mathcal{O}(-6) \end{array} \nwarrow (2V_3 \oplus V_3^\sharp) \otimes_k \mathcal{O}(-7) \longleftarrow V_1^\sharp \otimes_k \mathcal{O}(-8) \longleftarrow 0$$

or, numerically, of type

$$0 \longleftarrow \mathcal{I}_S \longleftarrow \begin{array}{c} \mathcal{O}(-5) \\ \oplus \\ 10\mathcal{O}(-6) \end{array} \nwarrow 15\mathcal{O}(-7) \longleftarrow 5\mathcal{O}(-8) \longleftarrow 0$$

We'll focus in the sequel on the properties of the new abelian surface of degree 15 and of the unique quintic hypersurface on which it lies.

To fix notations, assume as above the presentation matrix of M to be given by $\sigma_1 = \big(M_z(x) \mid M_{z_1}(x) \mid M_{z_2}(x)\big)$, where $z = (a_0, a_1, a_2, a_2, a_1) \in \mathbb{P}^2_+$, $z_1 = (0, 1, 0, 0, -1) \in \mathbb{P}^1_-$ and $z_2 = (0, 0, 1, -1, 0) \in \mathbb{P}^1_-$. In this setting the 10 linear syzygies of σ_1 are given by

$$\sigma_{21} = \begin{pmatrix} L_{y_1^-}(x) & L_{y_2^-}(x) \\ L_{y_{11}}(x) & L_{y_{21}}(x) \\ L_{y_{12}}(x) & L_{y_{22}}(x) \end{pmatrix},$$

where $L_y(x) = (x_{2i-j}y_{i-j})_{i,j}$ with $i, j \in \mathbb{Z}_5$, and the parameters have the following values

$$\begin{array}{lll} y_1^- = (0, 1, 0, 0, -1), & y_{11} = (2a_2, -a_1, 0, 0, -a_1), & y_{12} = (0, -a_2, a_0, a_0, -a_2), \\ y_2^- = (0, 0, 1, -1, 0), & y_{21} = (0, -a_0, a_1, a_1, -a_0), & y_{22} = (2a_1, 0, -a_2, -a_2, 0). \end{array}$$

However, the two quadratic syzygies involve only the last ten columns of σ_1. If $a_0 = 0$, then $M = \operatorname{coker} \sigma_1$ is not artinian. If $a_0 \neq 0$, which we'll assume in the sequel, they are given by the 15×2 matrix

$$\sigma_{22} = \begin{pmatrix} 0 & 0 \\ (x_{i+2}x_{i+3})_{i\in\mathbb{Z}_5} & (x_i^2)_{i\in\mathbb{Z}_5} \\ (x_i^2)_{i\in\mathbb{Z}_5} & (-x_{i+1}x_{i+4})_{i\in\mathbb{Z}_5} \end{pmatrix}.$$

Define now $\mathcal{G}$ as the kernel of the morphism ψ induced by the matrix with blocks $(\sigma_{21}, \sigma_{22})$

$$15\mathcal{O}(4) \xleftarrow{\psi} 10\mathcal{O}(3) \oplus 2\mathcal{O}(2) \longleftarrow \mathcal{G} \longleftarrow 0. \qquad (*)$$

For a general choice of the parameter $z \in \mathbb{P}^2_+$, $\mathcal{G}$ is a rank 2 reflexive sheaf because ψ has rank 10 (generically): it has rank smaller or equal 10 being defined by a part of the syzygies of σ_1, whereas it is easily seen that the submatrix of σ_{21} given by

$$A = \begin{pmatrix} L_{y_{11}}(x) & L_{y_{21}}(x) \\ L_{y_{12}}(x) & L_{y_{22}}(x) \end{pmatrix}$$

has a non-trivial determinant. In fact ψ has rank 10 outside codimension 3 since its restriction to a general $\mathbb{P}^2$, say $\{x_0 = x_1 = 0\}$ for $a_2 \neq 0$ or $\{x_0 = x_2 = 0\}$ for $a_1 \neq 0$, is an epimorphism on the restriction of $Syz_1(M)(5)$ to this plane. It follows that $\mathcal{G}$ has first Chern class $c_1 = -1$, and thus dualizing the exact sequence $(*)$ we obtain the minimal free resolution

$$0 \longleftarrow \mathcal{G} \longleftarrow \begin{array}{c} 2\mathcal{O}(-3) \\ \oplus \\ 10\mathcal{O}(-4) \end{array} \overset{\psi^T}{\nwarrow} 15\mathcal{O}(-5) \xleftarrow{\sigma_1^T} 5\mathcal{O}(-6) \longleftarrow 0.$$

In particular $\mathcal{G}$ is a stable rank 2 reflexive sheaf with Chern classes $c_1 = -1$, $c_2 = 9$, $c_3 = 25$ and $c_4 = 50$. Observe now that the morphism ψ drops rank on the line $E_{00} = \{x_1 - x_4 = x_2 - x_3 = a_0x_0 + 2a_1x_1 + 2a_2x_2 = 0\} \subset \mathbb{P}^2_+$, whence by Heisenberg invariance also on its translates $E_{ij} = \sigma^i\tau^j E_{00}$, for $i, j \in \mathbb{Z}_5$ by the group $\mathbb{H}_5$. Therefore $E_{ij} \subset \operatorname{Sing}(\mathcal{G})$, and since Porteus' formula yields $c_3(\mathcal{G}) = [\operatorname{Sing}(\mathcal{G})] = \deg \operatorname{Sing}(\mathcal{G})$ (see also [Ok3]) we see that the singular locus of the sheaf $\mathcal{G}$ consists of the 25 lines E_{ij}, $i, j \in \mathbb{Z}_5$. Now $\mathrm{h}^0(\mathcal{G}(3)) = 2$ and, by construction, the zero locus of section $s \in \mathrm{H}^0(\mathcal{G}(3))$ is a surface S of the type described before, thus a smooth abelian surface of degree 15 for a general choice of the parameter $z \in \mathbb{P}^2_+$ and of the section s:

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{G}(3) \longrightarrow \mathcal{I}_S(5) \longrightarrow 0.$$

Lemma 7.5. *The lines E_{ij}, $i, j \in \mathbb{Z}_5$, are the 25 exceptional lines of S.*

Proof. Clear since $\operatorname{Sing}(\mathcal{G})$ is contained in the zero locus of a section, while a line on an abelian surface is necessarily exceptional.□

One checks easily that $\mathcal{G}(3)$ is generated by sections outside the the lines E_{ij}. Therefore, for each fixed value of the parameter $z \in \mathbb{P}^2_+$, we obtain a pencil of abelian surfaces $S_{\lambda\mu} = \{\lambda s_1 + \mu s_2 = 0\} \subset \mathbb{P}^4$, where $\{s_1, s_2\}$ is a base of $\mathrm{H}^0(\mathcal{G}(3))$, all lying on the unique quintic hypersurface $V = \{s_1 \wedge s_2 = 0\} \subset \mathbb{P}^4$ and sharing the base locus of the pencil, namely the (-1) lines E_{ij}. To compute the equation of the quintic hypersurface we remark that from

$$\begin{pmatrix} M_{z_1}(x) & M_{z_2}(x) \end{pmatrix} \cdot A = -M_z(x) \cdot \begin{pmatrix} L_{y_1^-}(x) & L_{y_2^-}(x) \end{pmatrix}$$

it follows easily that $\det M_z(x) \mid \det A$ and thus finally that

$$V = \{\det A / \det M_z(x) = 0\}.$$

A more geometrical way to describe the quintic hypersurface V is given by the following

Lemma 7.6.
a) For a general choice of the parameter $z = (a_0, a_1, a_2, a_2, a_1) \in \mathbb{P}^2_+$, there exists an unique quintic hypersurface $V \subset \mathbb{P}^4$ containing the configuration $\cup_{i,j \in \mathbb{Z}_5} E_{ij}$. Furthermore, V is Heisenberg invariant.
b) For a general choice of the parameter z on the modular conic $C = \{a_0^2 + 4a_1a_2 = a_1 - a_4 = a_2 - a_3 = 0\} \subset \mathbb{P}^2_+$, the lines E_{ij}, $i, j \in \mathbb{Z}_5$, are rulings of an elliptic quintic scroll X, and thus are contained in five independent Segre cubics. Moreover, in this case, the homogeneous ideals I_X and $I_{\cup E_{ij}}$ coincide in degrees less or equal to 6.

Proof. We use the idea in [Au2, 2.2]. The group $\mathbb{H}_5$ acts as $\mathbb{Z}_5 \times \mathbb{Z}_5 = \mathbb{H}_5/Z(\mathbb{H}_5)$ on $\mathbb{P}^4$ and so $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(5)) = 6V_{0,0} \oplus_{(r,s)\neq(0,0)} 5V_{r,s}$, where $V_{r,s}$ are the characters of the group $\mathbb{Z}_5 \times \mathbb{Z}_5 =< \sigma > \times < \tau >$. On the other side, by construction, $\mathrm{H}^0(\mathcal{O}_{\cup E_{ij}}(5))$ is six times the regular representation, so, by Schur's lemma, the restriction morphism $\rho : \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(5)) \to \mathrm{H}^0(\mathcal{O}_{\cup E_{ij}}(5))$ decomposes as $\rho = \oplus \rho_{r,s}$, where $\rho_{0,0} : 6V_{0,0} \to 6V_{0,0}$, while $\rho_{r,s} : 5V_{r,s} \to 6V_{r,s}$ for $(r,s) \neq (0,0)$. As a consequence $\mathrm{H}^0(\mathcal{I}_{\cup E_{ij}}(5)) = \ker \rho = \oplus_{r,s} \ker \rho_{r,s}$.
Thus, in order to prove a), we check that the mappings $\rho_{r,s}$ are injective for $(r,s) \neq (0,0)$, while $\ker \rho_{0,0}$ is one dimensional. These are open conditions on $\mathbb{P}^2_+$, so it suffices to check them in a special case. Namely, we choose $a_0 = 1$, $a_1 = a_2 = 0$ and make the computations using explicit bases:

$$B_{r,s} = \{\sum_{i=0}^{4} \xi^{-ri} \prod_{j \in \mathbb{Z}_5}^{4} x_{i+j}^{m_j} \mid \sum_{j \in \mathbb{Z}_5} j m_j \equiv s \bmod 5, \sum_{j \in \mathbb{Z}_5} m_j = 5\}$$

for $5V_{r,s}$, when $(r,s) \neq (0,0)$, and the standard basis of $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(5))^{\mathbb{H}_5}$

$\gamma_0 = x_0x_1x_2x_3x_4$, $\gamma_1 = \sum_{i \in \mathbb{Z}_5} x_i x_{i+2}^2 x_{i+3}^2$, $\gamma_2 = \sum_{i \in \mathbb{Z}_5} x_i^3 x_{i+2} x_{i+3}$,

$\gamma_3 = \sum_{i \in \mathbb{Z}_5} x_i^3 x_{i+1} x_{i+4}$, $\gamma_4 = \sum_{i \in \mathbb{Z}_5} x_i x_{i+1}^2 x_{i+4}^2$, $\gamma_5 = \sum_{i \in \mathbb{Z}_5} x_i^5 - 5x_0x_1x_2x_3x_4$.

This is straightforward and we'll omit the details. We remark also that for this choice of the parameters $V = \{\gamma_0 = x_0x_1x_2x_3x_4 = 0\}$.

For part b), we observe that in this case E_{00} is tangent at z to the conic section invariant under the icosahedral group $\mathfrak{A}_5$ on $\mathbb{P}^2_+$: $C = \{x_0^2 + 4x_1x_2 = x_1 - x_4 = x_2 - x_3 = 0\}$. This conic can be naturally identified with the modular curve of level 5, which is in a

1:1 correspondence with the Heisenberg equivariantly embedded normal elliptic curves in $\mathbb{P}^4$ (see [BHM]). Under this identification E_{00} corresponds to the line spanned by two non-trivial 2-torsion points, say τ_1 and $\tau_1 + \tau_2$, on the elliptic curve E defined as the scheme-theoretic intersection of $Q_i = a_0 x_i^2 + 2a_1 x_{i+2} x_{i+3} + 2a_2 x_{i+1} x_{i+4} = 0$, $i \in \mathbb{Z}_5$. Then (cf. [BHM])

$$X = \bigcup_{p \in E} \overline{p, p + \tau_2} \subset \mathbb{P}^4$$

is a Heisenberg equivariantly embedded, elliptic quintic scroll containing the 25 rulings E_{ij}. X is embedded by a linear system numerically equivalent to $C_0 + 2f$, where f is the class of a ruling while C_0 is the class of a section with $C_0^2 = -e = 1$. Therefore, if there would exist a sextic hypersurface containing $\cup_{i,j \in \mathbb{Z}_5} E_{ij}$ and not containing the scroll, then its intersection with X would contain residual to the rulings E_{ij} an effective divisor whose numerical class is $6C_0 - 13f$, which is absurd.□

Part *b)* in the previous lemma shows that, for a choice of the parameter z on the conic C, the pencil of surfaces $S_{\lambda\mu}$ has the scroll X as base component because $I_{S_{\lambda\mu}}$ has generators in degrees 5 and 6. Moreover, it follows easily that in this case the quintic V is the trisecant variety of X (see [ADHPR] for its equation), thus it is singular and has multiplicity 3 along the scroll. Each $S_{\lambda\mu}$ is set-theoretically cut out by the quintic and two sextic hypersurfaces, so one checks easily that $S_{\lambda\mu}$ contains residual to the scroll a surface $T_{\lambda\mu}$ of degree 10 and sectional genus 6, which meets the scroll X along a section of degree 15. The general $T_{\lambda\mu}$ are smooth minimal abelian surfaces of degree 10, isogeneous to a product. A special member in the family is the first infinitesimal neighborhood of the elliptic quintic scroll X.

Remark 7.7. *For $z \in \mathbb{P}^2_+$ general, V coincides with the unique quintic hypersurface containing the abelian surface S. In terms of the above basis for $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(5))^{\mathbb{H}_5}$, V has the equation*

$$\begin{aligned}(a_0^5 - 8a_1^5 - 8a_2^5 + 15a_0^3 a_1 a_2)\gamma_0 + (a_0^4 a_1 + 8a_1^3 a_2^2 - 4a_0 a_2^4)\gamma_1 + (a_0^3 a_2^2 - 2a_0^2 a_1^3 - 4a_0 a_1 a_2^3)\gamma_2 \\ +(a_0^3 a_1^2 - 4a_0 a_1^3 a_2 - 2a_0^2 a_2^3)\gamma_3 + (a_0^4 a_2 + 8a_1^2 a_2^3 - 4a_0 a_1^4)\gamma_4 + a_0^3 a_1 a_2 \gamma_5 = 0\end{aligned}$$

Proof. First part is clear from (7.5) and (7.6), while for the second it is enough to check that

$$\begin{aligned}(a_0^5 - 8a_1^5 - 8a_2^5 + 15a_0^3 a_1 a_2)\gamma_0 + (a_0^4 a_1 + 8a_1^3 a_2^2 - 4a_0 a_2^4)\gamma_1 + (a_0^3 a_2^2 - 2a_0^2 a_1^3 - 4a_0 a_1 a_2^3)\gamma_2 \\ +(a_0^3 a_1^2 - 4a_0 a_1^3 a_2 - 2a_0^2 a_2^3)\gamma_3 + (a_0^4 a_2 + 8a_1^2 a_2^3 - 4a_0 a_1^4)\gamma_4 + a_0^3 a_1 a_2 \gamma_5 \in \ker \rho_{0,0}\end{aligned}$$

□

We fix from now on a general $z \in \mathbb{P}^2_+$.

Proposition 7.8. *The quintic V has 100 ordinary double points as singularities, four of them on each line E_{ij}.*

Proof. By computing, e.g., with [Mac], a standard basis for the jacobian ideal one checks that the singular locus of V is supported on $\cup_{i,j \in \mathbb{Z}_5} E_{ij}$. Moreover, if f denotes the above equation of V then

$$\frac{\partial f}{\partial x_2} - \frac{\partial f}{\partial x_3} \in I_{\mathbb{P}^2_+} \qquad a_0\left(\frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial x_3}\right) - 2a_2\frac{\partial f}{\partial x_0} \in I_{E_{00}}$$

$$\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_4} \in I_{\mathbb{P}^2_+} \qquad a_1\left(\frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial x_3}\right) - a_2\frac{\partial f}{\partial x_1} \in I_{E_{00}}$$

$$a_1\frac{\partial f}{\partial x_0} - a_0\frac{\partial f}{\partial x_1} \in I_{E_{00}}$$

thus, for $a_0 \neq 0$, the singularities of V on E_{00} are defined by $\frac{\partial f}{\partial x_0}$. Restricted to E_{00} $\frac{\partial f}{\partial x_0}$ has simple roots, and one checks in fact that, for general choices, these points are A_1 singularities on V. Therefore, by symmetry, V has 100 nodes as singularities. $\square$

Let Z denote the singular locus of V and let $\widehat{V}$ be a small resolution of the hypersurface; i.e., $\widehat{V}$ is smooth and a singular point $p \in Z$ is replaced by a $\mathbb{P}^1$, denoted by L_p. We recall that the defect of V is the rank of the subspace spanned by $\{L_p\}_{p\in Z}$ in $\mathrm{H}_2(\widehat{V}, \mathbb{Q})$ (cf. [We]). Also, by [Sch], it can be computed as

$$\text{defect}\,(V) = \mathrm{h}^1(\mathcal{I}_Z(5)).$$

We can write $Z = \cup_{g\in\mathbb{Z}_5\times\mathbb{Z}_5} \cup_{i=1}^4 g(p_i)$, where p_1, p_2, p_3 and p_4 are the singularities of V on E_{00}, and thus split the natural restriction morphism $\rho : \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(5)) \to \mathrm{H}^0(\mathcal{O}_Z)$ as $\rho = \oplus\rho_{r,s}$, with $\rho_{0,0} : 6V_{0,0} \to 4V_{0,0}$ and $\rho_{r,s} : 5V_{r,s} \to 4V_{r,s}$ for $(r,s) \neq (0,0)$. By (7.6) and (7.8) the mapping $\mathbb{P}^4 \dashrightarrow \mathbb{P}^5$ defined by $6V_{0,0}$ sends the points p_1, p_2, p_3 and p_4 to four distinct points spanning a $\mathbb{P}^4 \subset \mathbb{P}^5$. Hence $\ker\rho_{0,0}$, which comes from the set of hyperplanes through this $\mathbb{P}^3$, is 2-dimensional. As in [Au2, 2.2] we obtain:

Remark 7.9.
a) defect $(V) = 1$
b) Pic $(\widehat{V}) \cong \mathbb{Z}^2$ *, and* $e(\widehat{V}) = 200 - 2(\sharp\,\text{nodes}) = 0$
c) If $\widehat{H}$ is the pullback of a hyperplane section on V and $\widehat{S}$ the pullback of the abelian surface then $\{\widehat{H}, \widehat{S}\}$ is a basis for Pic $(\widehat{V})$
d) $\dim_{\mathbb{C}}\mathrm{H}^1(\widehat{V}, \Theta_{\widehat{V}}) = \mathrm{h}^1(\Omega^2_{\widehat{V}}) = \dim_{\mathbb{C}}(\mathrm{H}^0(\mathcal{I}_Z(5))/< x_i\frac{\partial f}{\partial x_j} >) = 2$*, thus $\mathbb{P}^2_+$ is the whole moduli space of these quintic hypersurfaces.*

Proof. Part *b)* follows from the fact that the Picard group of a small resolution of a nodal hypersurface in $\mathbb{P}^4$ is torsion free by Lefschetz' theorem, while its rank is the defect +1 (cf. [We]). For *d)*, one observes that $\oplus_{(r,s)\neq(0,0)} \ker\rho_{r,s}$ is contained in $< x_i\frac{\partial f}{\partial x_j} \mid 0 \leq i,j \leq 4 >$, so the claim follows from Griffith' residues.$\square$

Remark 7.10. *We recall that, in contrast with* (7.9)*, the general Horrocks-Mumford quintic hypersurfaces have also* 100 *nodes, but have bigger defect, namely* 3.

We mention in the sequel one further interesting degeneration of the abelian surface we've described above. Namely, for $a_0 = 1$, $a_1 = a_2 = 0$ the construction yields a pencil of (singular) surfaces $S_{\lambda,\mu} \subset \mathbb{P}^4$, $(\lambda,\mu) \in \mathbb{P}^1$, which all lie on the degenerated quintic

$V_0 = \{\gamma_0 = x_0x_1x_2x_3x_4 = 0\}$. Restricting the resolutions to each of the hyperplanes $H_i = \{x_i = 0\}$ we see that

$$S_{\lambda,\mu} = \bigcup_{i\in\mathbb{Z}_5} S_i(\lambda,\mu),$$

where $S_i(\lambda,\mu) \subset H_i$ are the cubic surfaces defined by

$$S_i(\lambda,\mu) = \{x_i = \lambda(x_{i+1}^2x_{i+3} - x_{i+2}x_{i+4}^2) + \mu(x_{i+1}x_{i+2}^2 - x_{i+3}^2x_{i+4}) = 0\}.$$

Fix now a general $(\lambda,\mu) \in \mathbb{P}^1$ and let $S_i = S_i(\lambda,\mu)$, for $i \in \mathbb{Z}_5$.

Proposition 7.11.
a) S_i is a smooth Del Pezzo surface in the hyperplane H_i. It is invariant under the action of τ, whereas $\sigma(S_i) = S_{i-1}$, $i \in \mathbb{Z}_5$.
b) Any two surfaces S_i and S_j meet along a smooth conic and a point outside it. There are altogether five such points, namely these being the vertices $p_i = \mathbb{P}(e_i)$, $i \in \mathbb{Z}_5$, of the standard simplex in $\mathbb{C}^5$. Through each point p_i pass exactly four such Del Pezzo surfaces.
c) Consider the complete pentagon with vertices p_i, $i \in \mathbb{Z}_5$, i.e., $C_\infty = \cup_{i\in\mathbb{Z}_5} L_i$, with $L_i = \{x_{i+2} = x_{i+3} = x_{i+4} = 0\}$ for $i \in \mathbb{Z}_5$, and $C_0 = \cup_{i\in\mathbb{Z}_5} L'_i$, with $L'_i = \{x_{i+1} = x_{i+3} = x_{i+4} = 0\}$. Then

$$S \cap H_i = S_i \cup L'_{i+1} \cup L_{i+2} \cup L_{i+3} \cup L'_{i+4}.$$

Furthermore, these four lines are exceptional on all S_j with $j \in \mathbb{Z}_5 \setminus \{i\}$.

Proof. The claims are straightforward using the explicit equations of S_i, so we omit the calculations.□

Observe now that the exceptional lines of the abelian surface degenerate in our situation to $E_{00} = \{x_0 = x_1 - x_4 = x_2 - x_3 = 0\}$ and $E_{ij} = \sigma^i\tau^j E_{00}$, with $i, j \in \mathbb{Z}_5$. One checks easily that E_{i0}, E_{i1}, E_{i2}, E_{i3} and E_{i4} are (-1) lines on the Del Pezzo surface S_i. As a consequence we obtain the following geometric characterization of our configuration:

Proposition 7.12. *For each $i \in \mathbb{Z}_5$, the lines E_{i0}, E_{i1}, E_{i2}, E_{i3}, E_{i4} and L_{i+3}, L'_{i+1} can be completed to a Schläfli's double six configuration of lines in the hyperplane H_i, which then determines the Del Pezzo surface S_i as the unique cubic surface in H_i containing the given double six.*

Proof. The intersection patterns are clear from the explicit description of the configuration, and the claim follows from [H-CV, §25].□

For comparison, we mention here a rather similar degeneration for the abelian surfaces of degree 15 which lie on three quintic hypersurfaces and thus come via liaison from degree 10. Namely, we start with the zero-scheme $Y_{\alpha\beta}$ of a special section of the twisted Horrocks-Mumford bundle $\mathcal{E}_{HM}(3)$, which is the union of 5 quadric surfaces [HM]:

$$Y_{\alpha\beta} = \bigcup_{i\in\mathbb{Z}_5} \{x_i = \alpha x_{i+2}x_{i+3} + \beta x_{i+1}x_{i+4} = 0\}.$$

It lies on the quintic V_0 and on two other independent quintic hypersurfaces, and thus can be linked on V_0 to a configuration

$$X_{\lambda\mu} = \bigcup_{i \in \mathbb{Z}_5} X_i(\lambda, \mu)$$

where this time

$$X_i(\lambda, \mu) = \{x_i = \lambda(x_{i+1}^2 x_{i+3} + x_{i+2} x_{i+4}^2) + \mu(x_{i+1} x_{i+2}^2 + x_{i+3}^2 x_{i+4}) = 0\},$$

the parameters λ, μ being functions of α, β and the linkage.

The configuration $X_{\lambda\mu}$ lies now on three quintic hypersurfaces, nevertheless has similar symmetry properties to those listed for $S_{\lambda,\mu}$ in proposition 7.11. Finally, there is an analogue of the claim (7.12), the only difference being that the role of the E_{ij} is taken here by the Horrocks-Mumford lines L_{ij}: $L_{00} = \mathbb{P}^1_-$ and $L_{ij} = \sigma^i \tau^j L_{00}$.

References

[Al1] Alexander, J.: Surfaces rationelles non-speciales dans $\mathbb{P}^4$. Math. Z. **200**, 87-110 (1988).

[Al2] Alexander, J.: Speciality one rational surfaces in $\mathbb{P}^4$. LNS **179**, 1-23, London Math. Soc., Cambridge Univ. Press (1992).

[ACGH] Arbarello, E., Cornalba, M., Griffiths, P.A., Harris, J.: Geometry of algebraic curves. Springer (1985).

[ADHPR] Aure, A., Decker, W., Hulek, K., Popescu, S., Ranestad, K.: The Geometry of Bielliptic Surfaces in $\mathbb{P}^4$ (preprint 1992).

[AF] Andreotti, A., Frankel, T.: The Lefschetz theorem on hyperplane sections. Ann. of Math. **69**, 713-717 (1959).

[Au1] Aure, A.: On surfaces in projective 4-space. Thesis, Univ. of Oslo (1987).

[Au2] Aure, A.: Surfaces on quintic threefolds associated to the Horrocks-Mumford bundle. LNM **1399**, 1-9 (1989).

[AR] Aure, A., Ranestad, K.: The Smooth Surfaces of Degree 9 in $\mathbb{P}^4$. LNS **179**, 32-46, London Math. Soc., Cambridge Univ. Press (1992).

[BB] Ballico, E., Bolondi, G.: The variety of module structures. Arch. der Math. **54**, 397-408 (1990).

[BBM] Ballico, E., Bolondi, G., Migliore, J.: The Lazarsfeld-Rao problem for liaison classes of two-codimensional subschemes of $\mathbb{P}^n$. Amer. J. of Math. (to appear).

[BF] Braun, R., Fløystad, G.: An upper bound for the degree of surfaces in $\mathbb{P}^4$ not of general type, Preprint Aarhus (1991).

[BOSS] Braun, R., Ottaviani, G., Schreyer, F.-O., Schneider, M.: Boundness for non-general type 3-folds in $\mathbb{P}^5$, to appear in Proceedings of CIRM Meeting, Analysis and Geometry, Trento (1991).

[BHM] Barth, W., Hulek, K., Moore, R. Degenerations of Horrocks-Mumford surfaces. Math. Ann. **277**, 735-755 (1987).

[BPV] Barth, W., Peters, C., Van de Ven, A.: Compact Complex Surfaces. Springer (1984).

[Br] Brivio, S.: Smooth Enriques surfaces in $\mathbb{P}^4$ and exceptional bundles, Preprint **201**, Genova (1992).

[Be] Beauville, A.: Complex algebraic surfaces. LNS **68**, London Math. Soc., Cambridge Univ. Press (1983).

[Bei] Beilinson, A.: Coherent sheaves on $\mathbb{P}^N$ and problems of linear algebra. Funct. Anal. Appl. **12**, 214-216 (1978).

[BM] Bolondi, G., Migliore, J.: The structure of an even liaison class. Trans. AMS. **316**, No. **1**, 1-38 (1989).

[Bo] Bombieri, E.: Methods of Algebraic geometry in char. p and their applications. C.I.M.E. , 59-95.

[BSS] Beltrametti, M., Schneider, M., Sommese A.J.: Threefolds of degree 11 in $\mathbb{P}^5$. LNS **179**, 59-80, London Math. Soc., Cambridge Univ. Press (1992).

[Ca] Catanese, F.: Surfaces with $K^2 = p_g = 1$ and their period mapping. Algebraic Geometry. Proceedings, Copenhagen (1978), LNM. **732**, 1-29 (1979).

[Ch] Chang, M. C.: Classification of Buchsbaum subvarieties of codimension 2 in projective space. J. Reine Angew. Math. **401**, 101-112 (1989).

[Cos] Cossec, F.: On the Picard group of Enriques surfaces. Math. Ann. **271**, 577-600 (1985).

[CV] Conte, A., Verra, A.: Reye constructions for nodal Enriques surfaces, Preprint Genova (1990).

[Deb] Debarre, O.: Inégalités numériques pour les surfaces de type général. Bull. Soc. math. France, **110**, 319-346 (1982).

[De1] Decker, W.: Stable rank 2 vector bundles with Chern-classes $c_1 = -1, c_2 = 4$. Math. Ann. **275**, 481-500 (1986).

[De2] Decker, W.: Monads and cohomology modules of rank 2 vector bundles. Comp. Math. **76**, 7-17 (1990).

[DES] Decker, W.,Ein, L., Schreyer, F.-O.: Construction of surfaces in $\mathbb{P}^4$. J. of Algebraic Geometry (to appear).

[E] Ein, L.: Surfaces with a hyperelliptic hyperplane section. Duke Math. J. **50**, 685-694 (1983).

[EP] Ellia, Ph., Peskine, Ch.: Groupes de points de $\mathbb{P}^2$; caractère et position uniforme. Algebraic Geometry. Proceedings, L'Aquila 1988, LNM. **1417**, 111-116 (1990).

[ES] Ellia, Ph., Sacchiero, G.: Surfaces lisses de $\mathbb{P}^4$ reglées en coniques, (talk at the conference "Projective Varieties", Trieste, June 1989).

[ElP1] Ellingsrud, G., Peskine, Ch.: Sur les surfaces lisses de $\mathbb{P}^4$. Inv. Math. **95**, 1-11 (1989).

[ElP2] Ellingsrud, G., Peskine, Ch.: Equivalence numérique pour les surfaces génériques d'une famille lisse de surfaces projectives. Problems in the theory of surfaces and their classification. Symposia Mathematica **XXXII**, 99-109 (1991).

[Fi] Finkelnberg, H.: Small resolutions of the Segre cubic. Indag. Math. **49**, 261-277 (1987).

[Fu] Fulton, W.: Intersection theory. Springer (1984).

[GH] Griffiths, P., Harris, J.: Principles of Algebraic Geometry. Wiley-Interscience (1978).

[GLP] Gruson, L., Lazarsfeld, R., Peskine, Ch.: On a theorem of Castelnuovo and equations defining space curves. Inv. Math. **72**, 491-506 (1983).

[GP1] Gruson, L., Peskine, Ch.: Genres des courbes de l'espace projectif. Algebraic Geometry Proceedings, Tromsø 1977, LNM. **687**, 31-59 (1978).

[GP2] Gruson, L., Peskine, Ch.: Genres des courbes de l'espace projectif (II). Ann. Scient. Ec. Norm. Sup., 4^{e} série, t.**15**, 401-418 (1982).

[Gr] Green, M.: Koszul Homology and the geometry of projective varieties. J. Diff. Geom. **19**, 125-171 (1984).

[Gro] Grothendieck, A.: Sur quelques points d'algèbre homologique. Tohoku Math. J. **9**, 119-221 (1957).

[Ha] Hartshorne, R.: Algebraic geometry. Springer (1977).

[H-CV] Hilbert, D., Cohn-Vossen, S.: Geometry and the Imagination. Chelsea Pub. Co., New York (1952).

[HM] Horrocks, G., Mumford, D.: A rank 2 vector bundle on $\mathbb{P}^4$ with 15000 symmetries. Topology **12**, 63-81 (1973).

[HR] Holme, A., Roberts, J.: On the embedding of projective varieties. Algebraic Geometry, Sundance 1986.Proceedings VI., LNM. **1311** (1988).

[Hu] Hulek, K.: Projective geometry of elliptic curves. Astérisque **137** (1986).

[HV] Hulek, K., Van de Ven, A.: The Horrocks-Mumford bundle and the Ferrand construction. Manuscripta Math. **50**, 313-335 (1985).

[Io1] Ionescu P.: Embedded projective varieties of small invariants I. Proc. of the week of Algebraic Geometry, Bucharest (1982), LNM. **1056** (1984).

[Io2] Ionescu P.: Embedded projective varieties of small invariants II. Rev. Roum. Math. Pures Appl. **31**, 539-544 (1986).

[Ja] James, C.G.F.: Extensions of a theorem of Segre's, and their natural position in space of seven dimensions. Proc. Camb. Phil. Soc. **21**, 664-684 (1923).

[Ka] Kalker, A.A.C.M.: Cubic fourfolds with fifteen ordinary double points. Thesis, Univ. of Leiden (1986).

[K] Kleiman, S: Geometry on grassmannians and applications to splitting bundles and smoothing cycles. IHES, Publ. Math. **36**, 282-298 (1969).

[Ko] Koelblen, L.: Surfaces de $\mathbb{P}^4$ tracées sur une hypersurface cubique. (preprint).

[La] Lanteri, A.: On the existence of scrolls in $\mathbb{P}^4$. Lincei-Rend. Sc. fis.mat.e.nat, Vol. **LXIX**, 223-227 (1980).

[LR] Lazarsfeld, R, Rao, P.: Linkage of general curves of large degree. Algebraic Geometry - open problems, Ravello 1982, LNM. **997**, 267-289, Springer 1983.

[LB] Le Barz, P.: Formules pour les multisecantes des surfaces. C.R. Acad. Sc. Paris, t.**292** Serie I, 797-799 (1981).

[Mac] Bayer, D., Stillman, M.: Macaulay: A system for computation in algebraic geometry and commutative algebra. Source and object code available for Unix and Macintosh computers. Contact the authors, or download from **zariski.harvard.edu** via anonymous ftp.

[Ma] Manolache N.: On the normal bundle to abelian surfaces embedded in $\mathbb{P}^4(\mathbb{C})$. Man. Math. **55**, 111-119 (1986).

[MDP] Martin-Deschamps, M., Perrin, D.: Sur la classification des courbes gauches. Astérisque **184-185** (1990).

[Mo] Moishezon, B.: Complex Surfaces and Connected Sums of Complex Projective Planes. LNM. **603** (1977).

[MR] Mezzetti, E., Ranestad, K.: The non-existence of a smooth sectionally non-special surface of degree 11 and sectional genus 8 in the projective fourspace. Manuscripta Math. **70**, 279-283 (1991).

[Ok1] Okonek, Ch.: Reflexive Garben auf $\mathbb{P}^4$. Math. Ann. **260**, 211-237 (1982).

[Ok2] Okonek, Ch.: Moduli reflexiver Garben und Flächen von kleinem Grad in $\mathbb{P}^4$. Math. Z. **184**, 549-572 (1983).

[Ok3] Okonek, Ch.: Über 2-codimensionale Untermannigfaltigkeiten vom Grad 7 in $\mathbb{P}^4$ und $\mathbb{P}^5$. Math. Z. **187**, 209-219 (1984).

[Ok4] Okonek, Ch.: Flächen vom Grad 8 in $\mathbb{P}^4$. Math. Z. **191**, 207-223 (1986).

[PS] Peskine, Ch., Szpiro, L.: Liaison des variétés algébriques I. Invent. Math. **26**, 271-302 (1974).

[PR] Popescu, S.E., Ranestad, K.: Surfaces of Degree 10 in the Projective Fourspace via Linear systems and Linkage. (to appear).

[R] Reider, I.: Vector bundles of rank 2 and linear systems on algebraic surfaces. Annals of Math. **127**, 309-316 (1988).

[Ra1] Ranestad, K.: On smooth surfaces of degree ten in the projective fourspace. Thesis, Univ. of Oslo (1988).

[Ra2] Ranestad, K.: Surfaces of degree 10 in the projective fourspace. Symp.Math. Vol. **XXXII**, 271-307 (1991).

[Ra3] Ranestad, K.: Private communication.

[Ro] Roth, L.: On the projective classification of surfaces. Proc. London Math. Soc. **42**, 142-170 (1937).

[Sch] Schoen, C.: Algebraic cycles on certain desingularized nodal hypersurfaces. Math. Ann. **270**, 17-27 (1985).

[Seg] Segre, C.: Sull' Incidenza di rette e piani nello spazio a quattro dimensioni. Rend. Palermo **II**, 42-52 (1888).

[Sem] Semple, J.G.: Note on rational normal quartic curves. J. London Math. Soc. **7**, 266-271 (1932).

[SR] Semple, J.G., Roth, L.: Introduction to algebraic geometry. Clarendon Press Oxford (1949).

[Sev] Severi, F.: Intorno ai punti doppi impropri di una superficie generale dello spazio a quattro dimensioni, e a suo punti tripli apparenti. Rend. Circ. Mat. Palermo **15**, 33-51 (1901).

[So] Sommese, A.J.: Hyperplane sections of projective surfaces I. The adjunction mapping. Duke Math. J. **46**, 377-401 (1979).

[SV] Sommese, A.J., Van de Ven, A.: On the adjunction mapping. Math. Ann. **278**, 593-603 (1987).

[VdV] Van de Ven, A.: On the 2-connectedness of very ample divisors on a surface. Duke Math. J. **46**, 403-407 (1979).

[W] Weil, A. Sur les critères d'équivalence en géométrie algebrique. Math. Ann. **128**, 95-127 (1954).

[We] Werner, J.: Kleine Auflösungen spezieller dreidimensionaler Varietäten. Thesis, M.P.I., Bonn (1987).

[Z] Zariski, O.: Algebraic Surfaces (Second supplemented Edition). Springer Verlag (1971).