Restricting linear syzygies: algebra and geometry

by

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Abstract: Let $X \subset \mathbb{P}^r$ be a closed scheme in projective space whose homogeneous ideal is generated by quadrics. In this paper we derive geometric consequences from the presence of a long strand of linear syzygies in the minimal free resolution of the ideal of X. These consequences are given in terms of the linear sections of X (the intersections of X with arbitrary linear subspaces).

More precisely, we say that X (or I_X) satisfies $\mathbf{N}_{2,p}$ if I_X has only linear syzygies for p steps. Thus X is 2-regular in the sense of Castelnuovo-Mumford iff it satisfies $\mathbf{N}_{2,p}$ for every $p \geq 1$. The simplest of our results says that if I_X is 2-regular, then the same is true for the ideal of any linear section of X, so long as the intersection has dimension ≤ 1. This is not in general true for higher-dimensional linear sections. We extend this result in a variety of ways, to projective subschemes satisfying $\mathbf{N}_{2,p}$ and to comparisons of resolutions of a subscheme and its linear sections. We use these results to bound homological invariants of some well-known projective varieties. In Eisenbud-Green-Hulek-Popescu [2004] we use some of the results of this paper to characterize and classify all 2-regular reduced projective schemes.

Extending a result of Fröberg [1990], we give a combinatorial characterization of the monomial ideals satisfying $\mathbf{N}_{2,p}$. Our results on the 2-regularity of sections yield a geometric characterization.

We also apply Green's "Linear Syzygy Theorem" [1999] to deduce a relation between the resolutions of I_X and $I_{X\cup\Gamma}$ for a scheme Γ, and apply the result to bound the number of intersection points of certain pairs of varieties such as rational normal scrolls.

Let V be a vector space of dimension $r+1$ over an algebraically closed field k with basis $x_0, \dots, x_r$. If $X \subset \mathbb{P}^r_k = \mathbb{P}(V)$ is a nondegenerate closed subscheme we write $\mathcal{I}_X$ for the ideal sheaf and I_X for the homogeneous ideal of X in the homogeneous coordinate ring $S = \mathrm{Sym}(V) = k[x_0, x_1, \dots, x_r]$ of $\mathbb{P}(V)$. Suppose that I_L is an ideal generated by linear forms, the ideal of a linear space L. In general there is no strong connection between the minimal free resolution of I_X and the minimal free resolution of $I_X + I_L$ or of its saturation. The goal of this paper is to exhibit some cases where an interesting connection of this kind exists.

If $X \subset \mathbb{P}^r$ is nondegenerate (that is, I_X contains no linear form) we say that X *satisfies the condition* $\mathbf{N}_{2,1}$ if I_X is generated by quadrics. We say that X *satisfies the condition* $\mathbf{N}_{2,p}$, for $p > 1$, if in addition the first p steps of the minimal free resolution

$$\cdots \longrightarrow F_t \xrightarrow{\phi_t} F_{t-1} \xrightarrow{\phi_{t-1}} \cdots \xrightarrow{\phi_1} F_0 \longrightarrow I_X \longrightarrow 0$$

of I_X are linear, in the sense that ϕ_t is represented by a matrix of linear forms for all $1 \leq t \leq p-1$ or, equivalently, that $\mathrm{Tor}_i^S(I_X, k)$ is a vector space concentrated in degrees $\leq i+2$ for all $i \leq p-1$. (Our notation comes from the notation $\mathbf{N}_p$ of Green and Lazarsfeld; but we do not insist that X be projectively normal, which is their condition $\mathbf{N}_0$ and is included in their condition $\mathbf{N}_p$. Also in some of our results we could replace degree 2 by an arbitrary degree d.)

In the first section of this paper we show that if $X \subset \mathbb{P}^r$ satisfies $\mathbf{N}_{2,p}$ then the same is true of $\Lambda \cap X$ for any linear subspace Λ such that $\dim \Lambda \cap X \leq 1$ and $\dim \Lambda \leq p$. It follows, for example, that $\deg(\Lambda \cap X)$ (or even the geometric degree) is then at most $\dim \Lambda - \dim(\Lambda \cap X) + 1$ (see Eisenbud-Green-Hulek-Popescu [2004], Theorem 2.2). Theorem 1.1 may be thought of as a generalization of the easy direction of Green's conjecture (Green [1984]) proved by Green and Lazarsfeld [1985], and gives a new proof of this result. Further, if $\dim \Lambda \leq p-1$, we show that the restriction map from the quadrics in $\mathbb{P}^r$ containing X to those in Λ containing $\Lambda \cap X$ is surjective. As an application we recover a version of a result of Vermeire [2001] on the linear system of quadrics through a variety satisfying property $\mathbf{N}_2$. If in addition X is linearly normal, $\dim \Lambda \cap X = 0$ and $\Lambda \cap X$ spans Λ, then the restriction of minimal free resolutions is surjective for $p-1$ steps. We give examples showing that these results are sharp in various senses.

To describe one of the implications of such results, we say that a closed subscheme $X \subset \mathbb{P}^r$ is *small* if every zero-dimensional linear section $\Lambda \cap X$ of X has $\deg(\Lambda \cap X) \leq 1 + \dim \Lambda$. If X is nondegenerate, reduced and irreducible, then it follows that $\deg X = 1 + \mathrm{cod}(X, \mathbb{P}^r)$. Such varieties "of minimal degree" were classified by Castelnuovo, Del Pezzo, and Bertini (rational normal curves, scrolls, the Veronese surface, etc). For reduced subschemes some cases were classified by by Xambò [1981], and the general case was recently done by us (see Eisenbud-Green-Hulek-Popescu [2004]) using Theorem 1.1 from this paper. In particular, we show there that small algebraic sets are all 2-regular. This yields the corollary that if a reduced subscheme $X \subset \mathbb{P}^r$ satisfies $\mathbf{N}_{2,p}$ for $p = \mathrm{cod}(X, \mathbb{P}^r)$, then X is actually 2-regular (Corollary 1.7).

In Section 2 we characterize property $\mathbf{N}_{2,p}$ for ideals generated by monomials. In the squarefree case, an ideal generated by quadratic squarefree monomials comes from a simplicial complex that is the clique complex of a graph, and the property $\mathbf{N}_{2,p}$ is determined by the length of the shortest cycle in the graph without a chord (Theorem 2.1; this result was suggested to us by Serkan Hoşten, Ezra Miller, and Bernd Sturmfels). A special case is Fröberg's result [1990] characterizing 2-regular square-free monomial ideals. The general monomial case could be reduced to the squarefree monomial case via polarization, but we give a direct analysis that yields a more precise result relating the property $\mathbf{N}_{2,p}$ for a monomial ideal to the corresponding property for the largest squarefree monomial ideal it contains (Proposition 2.4).

In a number of interesting cases a kind of strong converse to Theorem 1.1 holds: a subscheme $X \subset \mathbb{P}^r$ satisfies $\mathbf{N}_{2,p}$ if and only if every linear section $\Lambda \cap X$ of dimension zero satisfies $\deg(\Lambda \cap X) \le 1 + \dim \Lambda$. For example Green and Lazarsfeld [1988] prove that this is the case when X is a smooth linearly normal curve of degree $d \ge 3\,\mathrm{genus}(X) - 2$. (See also Eisenbud [2004] for an exposition and Eisenbud-Popescu-Schreyer-Walter [2002] for a different perspective.) One of the main results of our paper [2004] shows that this is also true for any reduced scheme when $p = \infty$. In Corollary 2.5 we show that this converse holds for any $p \ge 1$ when X is defined by a monomial ideal.

In Section 3 we use Theorems 1.1 and 1.2 of Section 1 to prove (conjecturally sharp) upper bounds for the property $\mathbf{N}_p$ for Veronese, Segre-Veronese, Plücker or Fano embeddings, as well as for certain embeddings of abelian varieties. In Section 4 we make use of the Eisenbud-Koh-Stillman conjecture (proved by Green [1999]) to analyze the restriction of linear syzygies to other simple subvarieties of the ambient space whose syzygies are known, e.g. rational normal curves, scrolls, Veronese surfaces, etc. As applications we give a new proof to Green's syzygetic Castelnuovo lemma and obtain bounds on the length of a zero-dimensional intersection of scrolls or Veronese surfaces (for the latter see also Eisenbud-Hulek-Popescu [2003]).

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1 Restricting syzygies to linear subspaces

In this section we show how the condition $\mathbf{N}_{2,p}$ influences low-dimensional linear sections.

Theorem 1.1 *Let $X \subset \mathbb{P}^r$ be a closed subscheme satisfying the property $\mathbf{N}_{2,p}$ with $p \ge 1$, and let $\Lambda \subset \mathbb{P}^r$ be a linear subspace of dimension $\le p$. If $\dim X \cap \Lambda \le 1$ then $\mathcal{I}_{X\cap\Lambda,\Lambda}$ is 2-regular. In particular, if $X \cap \Lambda$ is finite, then* $\mathrm{length}(X \cap \Lambda) \le \dim \Lambda + 1$.

The next two results strengthen Theorem 1.1 in different ways. The first is also proven, in a different way, in Eisenbud-Huneke-Ulrich [2004]. A special case of the first follows also from Caviglia [2003].

Theorem 1.2 *Let $X \subset \mathbb{P}^r$ be a closed subscheme satisfying the property $\mathbf{N}_{2,p}$ with $p \ge 1$, and let $\Lambda \subset \mathbb{P}^r$ be a linear subspace of dimension $\le p - 1$. If $\dim X \cap \Lambda = 0$, then the natural restriction $H^0(\mathcal{I}_X(2)) \to H^0(\mathcal{I}_{X\cap\Lambda,\Lambda}(2))$ is surjective.*

Theorem 1.3 *Let $X \subset \mathbb{P}^r$ be a closed subscheme satisfying the property* $\mathbf{N}_{2,p}$ *with $p \geq 1$, and let $\Lambda \subset \mathbb{P}^r$ be a linear subspace of dimension $\leq p$. If X is linearly normal, $X \cap \Lambda$ is zero-dimensional and $X \cap \Lambda$ spans Λ, then the natural restriction from the minimal free resolution of $\mathcal{I}_X$ to the minimal free resolution of $\mathcal{I}_{X\cap\Lambda,\Lambda}$ surjects on the first $p-1$ steps.*

Remark 1.4 The conditions in Theorems 1.1-1.3 are often sharp. Here are some examples:

1) The ideal $I \subset k[x_0, \dots, x_4]$ of 2×2-minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & 0 & x_2 \\ 0 & x_0 & x_1 & x_3 \end{pmatrix}$$

is saturated and defines a scheme $Y \subset \mathbb{P}^4$ consisting of a 2-plane with a certain multiplicity 3 embedded point. The scheme Y is a linear section of a 2-regular variety $X \subset \mathbb{P}^8$, the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$, which is 2-regular and thus satisfies $\mathbf{N}_{2,p}$ for every $p \geq 1$. If Y were at most 1-dimensional then we would conclude from Theorem 1.1 that I was 2-regular. However I is not even linearly presented. This shows that the hypothesis $\dim(X \cap \Lambda) \leq 1$ in Theorem 1.1 cannot be weakened.

2) The intersection of Y with the hyperplane $H = \{x_4 = 0\}$ is 1-dimensional, and thus 2-regular by Theorem 1.1. If Y were zero-dimensional we could conclude from Theorem 1.2 that the quadrics on H vanishing on $Y \cap H$ were all restrictions of quadrics on $\mathbb{P}^7$ vanishing on X. But the saturation J of $I + (x_4)/(x_4)$ has an extra quadratic generator. Thus the hypothesis $\dim(X \cap \Lambda) \leq 0$ in Theorem 1.2 cannot be weakened.

3) The homogeneous ideal

$$I = (x_0^2, x_0x_1, x_0x_2 - x_1x_4, x_0x_4, x_1x_2 - x_1x_4, x_2^2, x_2x_4) \subset k[x_0, \dots, x_4]$$

is saturated, satisfies $\mathbf{N}_{2,2}$, and defines a scheme $X \subset \mathbb{P}^4$ consisting of two lines concurrent at $p = (0:0:0:1:0) \in \mathbb{P}^4$ and having an embedded component at that point. The linear subspace $\Lambda = \{x_3 = x_4 = 0\}$ meets X in the simple point $q = (0:1:0)$ which does not span Λ. The truncation J in degrees ≥ 2 of the saturation of $I + (x_3, x_4)/(x_3, x_4)$ is thus 2-regular, but the natural restriction of linear syzygies between the minimal free resolutions of I and J is not surjective on Tor_1's. Thus the hypothesis $X \cap \Lambda$ spans Λ in Theorem 1.3 cannot be weakened.

4) The homogeneous ideal

$$I = (x_0^2, x_0x_1 - x_2x_4, x_0x_2 - x_2x_4, x_0x_3, x_0x_4, x_3x_4, x_4^2) \subset k[x_0, \dots, x_4]$$

is the saturated ideal of a 2-regular scheme $X \subset \mathbb{P}^4$ consisting of a 2-plane Π with two embedded points. Its restriction to the hyperplane $\{x_4 = 0\}$ (which contains the 2-plane) is a non-saturated ideal defining Π. Its saturation and truncation in degrees ≥ 2 is a 2-regular ideal $J \subset k[x_0, \dots, x_3]$, but the restriction map from the minimal free resolution of I to that of J is not onto. This shows that Theorem 1.3 is sharp.

Remark 1.5 Generalizing the condition $\mathbf{N}_{2,p}$ we say that a projective subscheme $X \subset \mathbb{P}^r$ *satisfies the condition* $\mathbf{N}_{d,p}$, for some $d \geq 2$, if $\mathrm{Tor}_t^S(I_X, k)$ is concentrated in degrees $\leq d+t$ for all $t \leq p-1$. For example, X satisfies $\mathbf{N}_{d,1}$ if I_X is generated in degrees $\leq d$ or, equivalently, if the truncation $(I_X)_{\geq d} = \oplus_{e \geq d} H^0(\mathcal{I}_X(e))$ of I_X in degrees $\geq d$ is generated in degree d. In general, it is easy to show that X satisfies $\mathbf{N}_{d,p}$ if and only if X satisfies $\mathbf{N}_{d,1}$ and the minimal free resolution $(I_X)_{\geq d}$ is linear for p steps in the sense above. The proofs below of Theorems 1.1 and 1.2 can be adapted to the case of ideals satisfying property $\mathbf{N}_{d,p}$, that is generated by forms of any degree d and having minimal free resolution with p linear steps (just replace in the twists by $\mathcal{O}_\Lambda(2)$ with twists of $\mathcal{O}_\Lambda(d)$).

For the proofs of all three theorems we use the hypercohomology spectral sequences for the complex obtained by restricting to Λ appropriate twists of the minimal free resolution of the ideal sheaf $\mathcal{I}_X$. To fix notations we recall that if

$$\mathcal{F}^\bullet : \quad \cdots \longrightarrow \mathcal{F}^{-m} \longrightarrow \mathcal{F}^{1-m} \longrightarrow \cdots \longrightarrow \mathcal{F}^{-1} \longrightarrow \mathcal{F}^0$$

is a complex on Λ, then its hypercohomology $\mathbf{H}(\mathcal{F}^\bullet)$ is computed by two spectral sequences associated to a Cartan-Eilenberg resolution (double complex) of $\mathcal{F}^\bullet$. The filtration by columns of the double complex induces a first spectral sequence with E_2 terms

$$'E_2^{i,j} = H^i(H^j(\Lambda, \mathcal{F}^\bullet)) \Longrightarrow \mathbf{H}^{i+j}(\mathcal{F}^\bullet),$$

while the filtration by rows induces a second spectral sequence with

$$''E_2^{i,j} = H^j(\Lambda, \mathcal{H}^i(\mathcal{F}^\bullet)) \Longrightarrow \mathbf{H}^{i+j}(\mathcal{F}^\bullet),$$

where $\mathcal{H}^m(\mathcal{F}^\bullet)$ denotes the m-th cohomology sheaf of the complex $\mathcal{F}^\bullet$.

We start with the case which is technically the simplest to handle:

Proof of Theorem 1.2: Let

$$\cdots \longrightarrow \mathcal{E}^{-n} \longrightarrow \mathcal{E}^{-n+1} \longrightarrow \cdots \longrightarrow \mathcal{E}^{-1} \longrightarrow \mathcal{E}^0 \longrightarrow \mathcal{I}_X \longrightarrow 0$$

be the sheafification of a minimal free resolution of the homogeneous ideal of X. We apply the spectral sequences above to the complex $\mathcal{F}^\bullet := \mathcal{E}^\bullet \otimes \mathcal{O}_\Lambda(2)$ obtained by restricting the resolution to Λ.

Using the fact that $X \cap \Lambda$ is zero-dimensional we first show that $\mathbf{H}^0(\mathcal{F}^\bullet) = H^0(\mathcal{I}_X \otimes \mathcal{O}_\Lambda(2))$. Since $\mathcal{E}^\bullet$ is a resolution, the sheaves $\mathcal{H}^i(\mathcal{F}^\bullet)$ for $i \leq -1$ have support on the zero-dimensional scheme $X \cap \Lambda$. Hence $H^j(\Lambda, \mathcal{H}^{-i}(\mathcal{F}^\bullet)) = 0$ for all $j \geq 1$ and $i \leq -1$. Thus the second hypercohomology spectral sequence degenerates at $''E_2$ and $''E_2^{i,-i} = 0$ for all $i \leq 0$. This shows that $\mathbf{H}^0(\mathcal{F}^\bullet) = {}''E_\infty^{0,0} = {}''E_2^{0,0}$. But $''E_2^{0,0} = H^0(\mathcal{I}_X \otimes \mathcal{O}_\Lambda(2))$ as required since $\mathcal{E}^\bullet$ is a resolution of $\mathcal{I}_X$.

We next use the hypothesis that X satisfies the $\mathbf{N}_{2,p}$ property to show that the natural restriction map from $H^0(\mathcal{I}_X(2))$ surjects onto $\mathbf{H}^0(\mathcal{F}^\bullet)$, which by the result of the previous paragraph is $H^0(\mathcal{I}_X \otimes \mathcal{O}_\Lambda(2))$. Consider for this the other spectral sequence. By hypothesis $\mathcal{F}^i$ is a direct sum of copies of $\mathcal{O}_\Lambda(i)$ for all $1-p \leq i \leq 0$. Since $\dim \Lambda \leq p-1$

$$'E_1^{i,j} = H^j(\Lambda, \mathcal{F}^i) = 0 \quad \text{for} \quad j \geq 1 \text{ and } -\dim \Lambda \leq i \leq 0.$$

In particular $\mathbf{H}^0(\mathcal{F}^\bullet) = {}'E_\infty^{0,0}$. Because $\mathcal{F}^i \neq 0$ only for $i \leq 0$ we see that ${}'E_1^{0,0}$ surjects via the natural map onto ${}'E_\infty^{0,0}$. On the other hand ${}'E_1^{0,0} = H^0(\Lambda, \mathcal{F}^0) = H^0(\mathcal{I}_X(2))$ since $\mathcal{E}^\bullet$ is the sheafification of the minimal free resolution of the homogeneous ideal of X. Combining these maps gives the desired surjection.

To complete the proof of the theorem we still need to show that the natural restriction map

$$H^0(\mathcal{I}_X \otimes \mathcal{O}_\Lambda(2)) \longrightarrow H^0(\mathcal{I}_{X\cap\Lambda,\Lambda}(2))$$

is surjective. Consider the short exact sequence

$$(*) \qquad 0 \longrightarrow \mathcal{I}_X \cap \mathcal{I}_\Lambda / \mathcal{I}_X \cdot \mathcal{I}_\Lambda \longrightarrow \mathcal{I}_X \otimes \mathcal{O}_\Lambda \longrightarrow \mathcal{I}_{X\cap\Lambda,\Lambda} \longrightarrow 0.$$

The kernel $\mathcal{K} := (\mathcal{I}_X \cap \mathcal{I}_\Lambda / \mathcal{I}_X \cdot \mathcal{I}_\Lambda)$ has support on the zero-dimensional scheme $X \cap \Lambda$ so $H^1(\mathcal{K}(2)) = 0$ and the surjectivity follows. ∎

Proof of Theorem 1.1: We use the same two hypercohomology spectral sequences, applied this time to each of the complexes $\mathcal{F}^\bullet = \mathcal{E}^\bullet \otimes \mathcal{O}_\Lambda(2-l)$ for $l \geq 1$. Using the spectral sequence ${}'E^{i,j}$ we first prove that $\mathbf{H}^l(\mathcal{F}^\bullet) = 0$. To see this note that for all i with $-(\dim \Lambda - l) \leq i \leq 0$ the term $\mathcal{F}^i$ is a direct sum of copies of $\mathcal{O}_\Lambda(-l+i)$. For such i we have ${}'E_1^{i,l-i} = H^{l-i}(\Lambda, \mathcal{F}^i) = 0$ and the required vanishing follows.

Next we use the second spectral sequence to show that

$$H^l(\mathcal{I}_X(2-l) \otimes \mathcal{O}_\Lambda) = {}''E_2^{0,l} = 0.$$

From $\mathbf{H}^l(\mathcal{F}^\bullet) = 0$ it follows that ${}''E_\infty^{0,l} = 0$. The vanishing of the terms

$${}''E_2^{i,-i+l+1} = H^{-i+l+1}(\Lambda, \mathcal{H}^i(\mathcal{F}^\bullet))$$

is automatic because $\mathcal{H}^i(\mathcal{F}^\bullet)$ is supported on $X \cap \Lambda$, which has dimension at most $1 < l+1 \leq -i+l+1$, for $i \leq 0$. This implies ${}''E_2^{0,l} = {}''E_\infty^{0,l} = 0$.

Finally, we show that $H^l(\mathcal{I}_{X\cap\Lambda,\Lambda}(2-l)) = 0$ by showing that $H^l(\mathcal{I}_X \otimes \mathcal{O}_\Lambda(2-l))$ surjects onto it. For this we use the short exact sequence $(*)$ twisted by $-l$. We see that it is enough to prove $H^{l+1}(\mathcal{K}(2-l)) = 0$. This is automatic because $l \geq 1 \geq \dim(X \cap \Lambda)$. ∎

Proof of Theorem 1.3: This time we use the spectral sequences on the complex $\mathcal{F}^\bullet = \mathcal{E} \otimes \Omega_\Lambda^{m+1}(m+2)$, with $0 \leq m \leq p-1$. Recall from Green [1984], or Green-Lazarsfeld [1988], or Eisenbud [2004] that if $Y \subset \mathbb{P}^m$ is a scheme with $H^1(\mathcal{I}_Y(1)) = 0$, then for all $m \geq 0$ we have

$$\mathrm{Tor}_m^S(I_Y, k)_{m+2} = H^1(\mathcal{I}_Y \otimes \Omega_{\mathbb{P}^m}^{m+1}(m+2))$$

where $S = S_{\mathbb{P}^m}$ is the homogeneous coordinate ring of $\mathbb{P}^m$. Since we have assumed that X is linearly normal we can apply this to $X \subset \mathbb{P}^r$. Since $X \cap \Lambda$ is 2-regular by Theorem 1.1, and $X \cap \Lambda$ spans Λ, we can also apply this with $Y = X \cap \Lambda$ and $\mathbb{P}^m = \Lambda$. This gives

$$\mathrm{Tor}_m^{S_\Lambda}(I_{X\cap\Lambda,\Lambda}, k)_{m+2} = H^1(\mathcal{I}_{X\cap\Lambda,\Lambda} \otimes \Omega_\Lambda^{m+1}(m+2)).$$

In the sequence $(*)$ the sheaf $\mathcal{K}$ has zero dimensional support, and we deduce that $H^1(\mathcal{I}_{X\cap\Lambda,\Lambda} \otimes \Omega_\Lambda^{m+1}(m+2)) = H^1(\mathcal{I}_X \otimes \Omega_\Lambda^{m+1}(m+2))$.

Now consider the spectral sequence $''E$. We have $''E_2^{i,j} = 0$ when $i < 0$ and $j > 0$. On the other hand we have

$$''E_2^{0,1} = \mathrm{Tor}_m^{S_\Lambda}(I_{X\cap\Lambda,\Lambda}, k)_{m+2}$$

by the argument above. For any $q \geq 2$ we have

$$''E_2^{0,q} = H^q(\Lambda, \mathcal{I}_X \otimes \Omega_\Lambda^{m+1}(m+2)).$$

These terms are equal to zero because the map

$$\mathcal{I}_X \otimes \Omega_\Lambda^{m+1}(m+2) \to \mathcal{O}_{\mathbb{P}^r} \otimes \Omega_\Lambda^{m+1}(m+2)$$

has zero-dimensional kernel and cokernel, and $H^q(\mathcal{O}_{\mathbb{P}^r} \otimes \Omega_\Lambda^{m+1}(m+2)) = 0$. This shows that

$$\mathbf{H}^1(\mathcal{F}^\bullet) = \mathrm{Tor}_m^{S_\Lambda}(I_{X\cap\Lambda,\Lambda}, k)_{m+2}.$$

Next we turn to $'E$. We have $'E_1^{i,j} = H^j(\Lambda, \mathcal{E}^i \otimes \Omega_\Lambda^{m+1}(m+2))$. If $0 < j < \dim\Lambda$, then Bott's formula gives $'E_1^{i,j} = 0$ unless $j = m+1$ and $i = -m$. Because X satisfies property $\mathbf{N}_{2,p}$ and $m \leq p-1$, we get $\mathcal{E}^{-m} = \mathrm{Tor}_m^S(I_X, k)_{m+2} \otimes \mathcal{O}_{\mathbb{P}^r}(-m-2)$ so

$$'E_1^{-m,m+1} = H^{m+1}(\Lambda, \mathcal{E}^{-m} \otimes \Omega_\Lambda^{m+1}(m+2)) = \mathrm{Tor}_m^S(I_X, k)_{m+2}.$$

On the other hand if $i \geq -\dim\Lambda + 1$ then $'E_1^{i,\dim\Lambda} = H^{\dim\Lambda}(\Lambda, \mathcal{E}^i \otimes \Omega_\Lambda^{m+1}(m+2)) = 0$. Thus $'E_1^{-m,m+1}$ surjects onto

$$'E_\infty^{-m,m+1} = \mathbf{H}^1(\mathcal{F}^\bullet) = \mathrm{Tor}_m^{S_\Lambda}(I_{X\cap\Lambda,\Lambda}, k)_{m+2}.$$

This is the natural map induced by the surjection $I_X \to I_{X\cap\Lambda,\Lambda}$. ∎

Recall from the introduction (see also Eisenbud-Green-Hulek-Popescu [2004]) that a closed non-degenerate subscheme $X \subset \mathbb{P}^r$ is called "small" if every zero-dimensional linear section of X imposes independent conditions on linear forms, or equivalently if every zero-dimensional linear section of X is 2-regular.

In this language, we may rephrase Theorem 1.1 as follows:

Theorem 1.6 *Let $X \subset \mathbb{P}^r$ be a closed subscheme which satisfies property $\mathbf{N}_{2,p}$, for some $p \geq 1$. If $\Lambda \subset \mathbb{P}^r$ is a linear subspace such that $\mathrm{cod}(X\cap\Lambda, \mathrm{span}(X\cap\Lambda)) \leq p-1$, then $\Lambda \cap X$ is small. In particular, 2-regular projective schemes are small.* ∎

In Eisenbud-Green-Hulek-Popescu [2004] we prove that small and 2-regular projective algebraic sets are the same. In particular, we obtain the following unexpected consequence of Theorem 1.6:

Corollary 1.7 *If $X \subset \mathbb{P}^r$ is a reduced subscheme satisfying property $\mathbf{N}_{2,p}$ for $p = \operatorname{cod}(X, \mathbb{P}^r)$, then X is 2-regular.* ∎

For a geometric description and the classification of such reduced schemes see Eisenbud-Green-Hulek-Popescu [2004]).

As an immediate application of Theorem 1.2 we get an easy geometric proof of the known direction of Green's conjecture proved by Green and Lazarsfeld in [1985].

Corollary 1.8 *Let C be a non-hyperelliptic curve of genus $g \geq 3$, and let $\nu = \operatorname{Cliff}(C)$. Then the canonical embedding of C does not satisfy property $\mathbf{N}_\nu$.*

Proof. Let $\mathcal{O}(L) \in \operatorname{Pic}(C)$ be a degree d line bundle that realizes the Clifford index of C with $|L|$ and $|K-L|$ base-point free series. Choose D_1 and D_2 divisors in $|L|$ and $|K-L|$, respectively, consisting of distinct points and such that $D_1 \cap D_2 = \emptyset$. Since D_1 and D_2 add up to K their union spans only a hyperplane and thus the geometric Riemann-Roch yields that $\Lambda := \operatorname{span}(D_1) \cap \operatorname{span}(D_2)$ has dimension $\nu - 1$. On the other hand, $\operatorname{span}(D_i)$ meets the canonical curve only along the points of D_i, $i = 1, 2$, otherwise C would have a lower Clifford index, and therefore $\Lambda \cap C = \emptyset$. Thus from Theorem 1.2, if C satisfies property $\mathbf{N}_\nu$, we deduce that $H^0(\mathcal{I}_C(2)) \longrightarrow H^0(\mathcal{O}_\Lambda(2))$ must be surjective.

Let D be a divisor of degree $\leq 2g-2$. From the cohomology sequence

$$\cdots \longrightarrow H^0(\mathcal{O}(2K)) \longrightarrow H^0(\mathcal{O}_D(2K)) \longrightarrow H^1(\mathcal{O}(2K-D)) \longrightarrow \cdots$$

we see that D fails to impose independent conditions on quadrics in $\mathbb{P}^{g-1}$ if and only if $D \in |K|$. In particular, D_1 and D_2 impose independent conditions on quadrics, however $D_1 + D_2$ fails by one to impose independent conditions on quadrics. This leads now to a contradiction since the surjectivity of $H^0(\mathcal{I}_C(2)) \longrightarrow H^0(\mathcal{O}_\Lambda(2))$, together with the fact that D_i, $i = 1, 2$, impose independent conditions on quadrics implies that their sum $D_1 + D_2$ would also impose independent conditions. ∎

We can also derive a result of Vermeire [2001] on rational mappings of projective space:

Corollary 1.9 *If $X \subset \mathbb{P}^r$ satisfies $\mathbf{N}_{2,2}$ and $\operatorname{Sec}(X) \neq \mathbb{P}^r$, then the linear system $|H^0(\mathcal{I}_X(2))|$ on $\mathbb{P}^r$ is one-to-one outside of $\operatorname{Sec}(X)$.*

Proof. Let $x_1, x_2 \in \mathbb{P}^r \setminus X$ be a pair of points imposing only one condition on the quadrics of $|H^0(\mathcal{I}_X(2))|$ and let $\Lambda = \overline{x_1, x_2}$ be the line they span. By Theorem 1.2 the restriction map $H^0(\mathcal{I}_X(2)) \to H^0(\mathcal{I}_{X \cap \Lambda, \Lambda}(2))$ is surjective and thus Λ must be a secant line of $X \subset \mathbb{P}^r$. ∎

The following corollary may be regarded as a generalization of Corollary 1.9 for the case when property $\mathbf{N}_p$, $p \geq 2$ holds:

Corollary 1.10 *Let $X \subset \mathbb{P}^r$ be a closed subscheme satisfying the property $\mathbf{N}_{2,p}$ for some $p \geq 2$, and let $x_1, \ldots, x_p \in \mathbb{P}^r \setminus X$ be points in linearly general position which fail to impose independent conditions on the quadrics containing X. Let $\Lambda \cong \mathbb{P}^{p-1}$ be the linear span of $\{x_1, \ldots, x_p\}$ and assume that $\Lambda \cap X$ is zero-dimensional and reduced. Then for some $2 \leq q \leq p$ there exist subsets $Z_1 \subset \{x_1, \ldots, x_p\}$ and $Z_2 \subset \Lambda \cap X$, both of cardinality q, such that $Z_1 \cup Z_2$ spans a $\mathbb{P}^{q-1}$ and fails (exactly by one) to impose independent conditions on quadrics in $\mathbb{P}^{q-1}$ (in other words $Z_1 \cup Z_2$ is self-associated).*

Proof. By Theorem 1.2 the restriction map $H^0(\mathcal{I}_X(2)) \to H^0(\mathcal{I}_{X\cap\Lambda,\Lambda}(2))$ is surjective, so the hypothesis means that the points $x_1, \ldots, x_p \in \Lambda$ fail to impose independent conditions on the quadrics in $|H^0(\mathcal{I}_{X\cap\Lambda,\Lambda}(2))|$. On the other hand, by Theorem 1.1 we know that $\deg(X \cap \Lambda) \leq p$. The conclusion follows now from a result of Dolgachev-Ortland [1988] (Lemma 3, p. 45) and Shokurov [1971] which implies that every subscheme of $\Gamma := (\Lambda \cap X) \cup \{x_1, \ldots, x_p\} \subset \Lambda$ of degree $\leq 2p$ does impose independent conditions on quadrics in Λ if no subset of $2s+2 < 2p+2$ points of Γ is contained in a $\mathbb{P}^s$. (See Eisenbud-Popescu [2000] for the connection with self-association and the Gorenstein property.) ∎

2 Monomial ideals satisfying $\mathbf{N}_{2,p}$

In this section we analyze the conditions $\mathbf{N}_{2,p}$ for monomial ideals. We shall see that in the saturated case (and somewhat more generally) Theorem 1.1 provides a criterion to decide which of these conditions are satisfied.

We begin with the case of squarefree monomial ideals. Using the Stanley-Reisner correspondence, a squarefree monomial ideal $I \subset S = k[x_0, \ldots, x_r]$ corresponds to a simplicial complex $\Delta(I)$ with vertices the variables of the ring S (see for instance Stanley [1996] for details). We will denote by I_Δ the Stanley-Reisner ideal corresponding to a simplicial complex Δ, and for simplicity we will assume that no variable x_i is among the minimal generators of I_Δ.

Recall that if G is a graph, then a *clique* of G is a subset T of vertices of G such that G contains every edge joining two vertices of T. The *clique complex* or *flag complex* of G is the simplicial complex $\Delta(G)$ whose faces are precisely the cliques of G; the graph G is the 1-skeleton of the clique complex of G.

Clique complexes occur frequently: For any poset P its order complex $\Delta(P)$ (the complex whose faces are the chains of P) is the clique complex $\Delta(G_P)$ of the comparability graph of P (the graph on the vertices of P whose edges are all pairs of comparable vertices). In particular the barycentric subdivision of any simplicial complex is a clique complex.

It is easy to see that a simplicial complex Δ is a clique complex if and only if every minimal non-face of Δ consists of 2 vertices. Thus I_Δ is generated by quadratic monomials if and only if Δ is a clique complex (of its 1-skeleton), so to study the properties $\mathbf{N}_{2,p}$ we restrict ourselves to clique complexes.

The following result was suggested to us by Serkan Hoşten, Ezra Miller, and Bernd Sturmfels. A *cycle* C in G of length q is a sequence of distinct edges of G of the form $(v_1, v_2), (v_2, v_3), \ldots, (v_q, v_1)$ joining distinct vertices $v_1, \ldots, v_q$, for some $q \geq 3$. We say that the cycle C has a *chord* if some (v_i, v_j) is an edge of G, with $j \not\equiv i+1 \pmod q$. We say that the cycle is *minimal* if $q > 3$ and C has no chord. Thus the first homology group of $\Delta(G)$ is generated by minimal cycles.

Theorem 2.1 *Let $I = I_\Delta$ be an ideal generated by quadratic squarefree monomials, and let G be the 1-skeleton of Δ. The ideal I satisfies the condition $\mathbf{N}_{2,p}$, with $p > 1$, if and only if every minimal cycle in G has length $\geq p+3$.*

As a special case we obtain the main theorem of Fröberg [1990]. We say that G is *chordal* if every cycle of length > 3 has a chord; in other words if G has no minimal cycles.

Corollary 2.2 *A square-free monomial ideal I_Δ is 2-regular if and only if Δ is the clique complex of a chordal graph.* ∎

The proof of Theorem 2.1 makes use of Reisner's Theorem (see for example Hochster [1977], or Stanley [1996]): If $I_\Delta \subset S = k[x_0, \ldots, x_r]$ is a squarefree monomial ideal corresponding to the simplicial complex Δ, then $\operatorname{Tor}_i^S(I_\Delta, k)$ is a $\mathbb{Z}^{r+1}$-graded vector space which is nonzero only in degrees corresponding to squarefree monomials m and

$$\operatorname{Tor}_i^S(I_\Delta, k)_m = \widetilde{H}_{\deg(m)-i-2}(|m|, k),$$

where $\widetilde{H}_i(|m|, k)$ denotes the i-th reduced homology of the full subcomplex $|m|$ of Δ whose vertices correspond to the variables dividing m.

Example 2.3 For example, let $d \geq 3$ be an integer. If Δ is the simplicial complex with $d+1$ vertices and $d+1$ edges forming a simple cycle, then the reduced homology of any full proper subcomplex of Δ is concentrated in degree 0, while the reduced homology of the empty set is in degree -1 and the reduced homology of Δ itself is k, concentrated in degree 1. We deduce that the minimal free resolution of S/I_Δ has the form

$$0 \longrightarrow S(-d-1) \longrightarrow S(-d+1)^{\beta_{d-2}} \longrightarrow \cdots \longrightarrow S(-2)^{\beta_1} \longrightarrow S,$$

so that I_Δ satisfies $\mathbf{N}_{2,d-2}$, but not $\mathbf{N}_{2,d-1}$. Further, the algebraic set X defined by I_Δ consists of $d+1$ lines joined in a cycle in $\mathbb{P}^d$. The ring $S_X = S/I_\Delta$ is Cohen-Macaulay, even Gorenstein, and X is a curve of degree $d+1$ and arithmetic genus 1 — a degenerate *elliptic normal curve*. If Λ is the hyperplane defined by the vanishing of the sum of the variables (or any hyperplane not containing one of the components of X), then $\Lambda \cap X$ is a set of $d+1$ points in a $(d-1)$-dimensional plane, and is thus not 2-regular.

Proof of Theorem 2.1. Let $x_0, \ldots, x_r$ be the vertices of G, and write $S = k[x_0, \ldots, x_r]$ for the ambient polynomial ring. Let X be the algebraic set defined by I_Δ in $\mathbb{P}^r$.

First assume that G has a minimal cycle C of length $p + 2 > 3$. Let J be the ideal generated by the variables not in the support of C, and let Λ be the projective linear subspace in $\mathbb{P}^r$ defined by J. The plane section $\Lambda \cap X \subset \Lambda$ has homogeneous coordinate ring $S/(I_\Delta + J) = S'/I_C$ where $S' = S/J$. As we showed in the example above, the ideal I_C is not 2-regular. By Theorem 1.1, the ideal I_Δ does not satisfy $\mathbf{N}_{2,p}$. (Of course the same result may be proven by applying Reisner's Theorem directly to Δ, by taking $|m| = C$.)

Conversely, suppose that I does not satisfy the condition $\mathbf{N}_{2,p}$, and take $p > 1$ minimal with this property. We must show that Δ contains a minimal $(p+2)$-cycle.

By Reisner's Theorem there exists a squarefree monomial m of minimal degree $\deg(m) \geq p+2$ such that $\widetilde{H}_{\deg(m)-p-1}(|m|, k) \neq 0$, while $\widetilde{H}_{\deg(m')-i-2}(|m'|, k) = 0$ for all $0 \leq i \leq \min(p-1, \deg(m') - 3)$ and all $m'|m$ with $m' \neq m$. If $\deg(m) = p+2$, then $\widetilde{H}_1(|m|, k) \neq 0$ or equivalently the edge-path group of the simplicial complex $|m|$ is not trivial. Since m is of minimal degree with the above property, the simplicial complex $|m|$ must be connected, and again minimality and the fact that Δ is a clique complex imply that $|m|$ consists of a cycle of length $p + 2$ in G, and this cycle is minimal (see also Spanier [1966, Theorem 3, p.140] for a description by generators and relation of the edge-path group). This is exactly the claim of the theorem.

If however $\deg(m) > p+2$, let $m'|m$ be a squarefree monomial with $\deg(m') = \deg(m) - 1$ and denote by x the extra variable in the support of m. There is a long exact sequence

$$\ldots\ \widetilde{H}_i(|m'|, k) \longrightarrow \widetilde{H}_i(|m|, k) \longrightarrow \widetilde{H}_{i-1}(\mathrm{link}(x, |m|), k) \longrightarrow \widetilde{H}_{i-1}(|m'|, k)\ \ldots.$$

which is obtained from the long exact homology sequence of the pair $(|m|, |m'|)$ and the isomorphisms

$$\widetilde{H}_i(|m|, |m'|, k) \quad \cong \quad \widetilde{H}_i(\mathrm{star}(x, |m|), \mathrm{link}(x, |m|), k) \quad \cong \quad \widetilde{H}_{i-1}(\mathrm{link}(x, |m|), k)$$

for all i. The last isomorphism comes from the long exact sequence of the second pair which breaks up into isomorphisms since $\mathrm{star}(x, |m|)$ is contractible.

Since $\widetilde{H}_{\deg(m)-p-1}(|m|, k) \neq 0$ while $\widetilde{H}_{\deg(m)-p-1}(|m'|, k) = 0$, we deduce from the long exact sequence that $\widetilde{H}_{\deg(m)-p-2}(\mathrm{link}(x, |m|), k) \neq 0$, with $\deg(m) - p - 2 \geq 1$. On the other hand the simplicial complex $\mathrm{link}(x, |m|)$ is a full (strict) subcomplex of $|m|$ and thus of Δ. Indeed if $x_{i_1}, \ldots, x_{i_s} \in \mathrm{link}(x, |m|)$ are vertices such that $\{x_{i_1}, \ldots, x_{i_s}\} \in |m| \subseteq \Delta$, then obviously $\{x_{i_a}, x_{i_b}\} \in \Delta$ for all $a \neq b$, and also $\{x, x_{i_a}\} \in \Delta$ by the definition of the link. Since Δ is a clique complex it follows that $\{x, x_{i_1}, \ldots, x_{i_s}\}$ must also be a face of Δ with support in $|m|$. But this means that we have found a full subcomplex $|m''| = \mathrm{link}(x, |m|)$ of Δ, with $\deg(m'') < \deg(m)$, such that $\widetilde{H}_j(|m''|, k) \neq 0$ for some $j \geq 1$, which contradicts the fact that I_Δ satisfies property $\mathbf{N}_{2,p-1}$. This concludes the proof of the theorem. ∎

As Fröberg remarks, the case of a general ideal $I \subset S = k[x_0, \ldots, x_r]$ generated by quadratic monomials may be reduced, by the process of polarization, to the squarefree case. However, we can give a more explicit result. We may harmlessly assume that I contains no linear forms, and we may write I uniquely in the form $I = I_\Delta + I_s$ for some simplicial complex Δ with vertices $x_0, \ldots, x_r$ and where the ideal I_s is generated by $\{x_i^2 \mid x_i^2 \in I\}$. We will refer to the vertices x of Δ such that $x^2 \in I$ as the *square vertices for I*.

Proposition 2.4 *Let $I = I_\Delta + I_s$ be an ideal generated by quadratic monomials, decomposed as above.*

a) The ideal I satisfies $\mathbf{N}_{2,2}$ if and only if I_Δ satisfies $\mathbf{N}_{2,2}$ and for any square vertex x for I, $\mathrm{link}(x, \Delta)$ is a simplex not containing any square vertex for I.

b) If I satisfies $\mathbf{N}_{2,2}$, then I satisfies $\mathbf{N}_{2,p}$ for some $p \geq 3$ if and only if I_Δ satisfies $\mathbf{N}_{2,p}$.

Proof. If $I_s = (0)$ the result is obvious. Otherwise, let x be a square vertex for I, and let $I' = I_\Delta + I'_s$, where $I'_s \subset I_s$ is the ideal generated by the squares of all square vertices for I other than x. The exact sequence

$$0 \longrightarrow ((I' : x^2)/I')(-2) \longrightarrow S/I'(-2) \xrightarrow{x^2} S/I' \longrightarrow S/I \longrightarrow 0$$

and the observation that $(I' : x^2) = (I' : x)$ yields a short exact sequence

$$0 \longrightarrow (S/(I' : x))(-2) \xrightarrow{x^2} S/I' \longrightarrow S/I \longrightarrow 0.$$

From the long exact sequence in Tor's, we see that I satisfies property $\mathbf{N}_{2,2}$ if and only if I' satisfies $\mathbf{N}_{2,2}$ and $(I' : x)$ is generated by linear forms. On the other hand, we have $(I' : x) = I_{\mathrm{link}(x,\Delta)} + I'_s$. This is generated by linear forms if and only if $\mathrm{link}(x, \Delta)$ is a simplex not containing any of the square vertices that appear in I'_s, as required. This proves part *a*).

When $(I' : x)$ is generated by linear forms, each $\mathrm{Tor}_i^S(S/(I' : x), k)$ is concentrated in degree i. In this circumstance the long exact sequence in Tor's coming from the short exact sequence above shows that I satisfies $\mathbf{N}_{2,p}$ for some $p \geq 3$ if and only if I' satisfies $\mathbf{N}_{2,p}$, and we are done by induction. ∎

Corollary 2.5 *If $I = I_X$ is the ideal of a closed subscheme $X \subset \mathbb{P}^r$, and I is generated by quadratic monomials, then I satisfies $\mathbf{N}_{2,p}$ if and only if for all planes Λ of dimension $\leq p$ having zero-dimensional intersection with X the scheme $\Lambda \cap X$ is 2-regular.*

We first need a characterization of saturated ideals:

Lemma 2.6 *Let $I = I_\Delta + I_s$ be an ideal generated by quadratic monomials, decomposed as above, with I_Δ a squarefree quadratic monomial ideal and I_s the ideal generated by the squares of the square vertices for I. Then I is saturated if and only if every maximal face of Δ contains at least one non-square vertex for I.*

Proof. If the ideal generated by all the vertices is associated, it must annihilate a squarefree monomial, and this must be the product of all vertices in a facet of Δ. Such a product is annihilated by the maximal ideal if and only if every vertex in that facet is a square vertex for I. ∎

Proof of Corollary 2.5. If a linear subspace Λ of dimension $\leq p$ meets X in a zero-dimensional scheme $X \cap \Lambda$ that is not 2-regular, then Theorem 1.1 shows that I does not satisfy $\mathbf{N}_{2,p}$.

Conversely, suppose that I does not satisfy $\mathbf{N}_{2,p}$, with $p \geq 2$ minimal, and decompose $I = I_\Delta + I_s$ as above. If $p > 2$ then from Proposition 2.4 *b*) we see that I_Δ does not satisfy $\mathbf{N}_{2,p}$, and thus the 1-skeleton of Δ has a minimal cycle C of length $p + 2$. If x is a vertex of such a cycle then $\mathrm{link}(x, \Delta)$ is not a simplex, and it follows that x is not a square vertex for I. If Λ' is the linear subspace spanned by all the vertices in the cycle C, then $X \cap \Lambda' \subset \Lambda'$ is a degenerate "elliptic normal curve" as in Example 2.3. As remarked in that example, any sufficiently general plane $\Lambda \subset \Lambda'$ of codimension 1 in Λ' is a p-plane that meets X in a zero-dimensional scheme that is not 2-regular.

Finally, suppose that I does not satisfy $\mathbf{N}_{2,2}$. We use the characterization in part *a*) of Proposition 2.4. If I_Δ does not satisfy $\mathbf{N}_{2,2}$ then we proceed as before. Otherwise there is a square vertex x for I such that either the link of x in Δ is not a simplex, or the link of x in Δ is a simplex containing another square vertex for I.

In the first case we can choose vertices y, z in $\mathrm{link}(x, \Delta)$ such that $yz \in I_\Delta \subseteq I$. Factoring out all the variables except x, y, z we get from I a monomial ideal $\overline{I}$ with

$$(x^2, yz) \subset \overline{I} \subset (x^2, y^2, yz, z^2) \subset k[x, y, z].$$

Any such ideal is saturated, so it defines the zero-dimensional scheme $X \cap \Lambda \subset \Lambda$, where Λ is the 2-plane spanned by the vertices x, y, z. Further, $\overline{I}$ is not 2-regular.

In the second case, let y be one of the square vertices for I such that $y \in \mathrm{link}(x, \Delta)$. Since the link of x is a simplex, the star of x (that is, x together with the link) is a maximal face of Δ. Since I is saturated we may choose a vertex $z \in \mathrm{link}(x, \Delta)$ that is not a square vertex for I. Factoring out all the variables except x, y and z we get from I the saturated ideal $\overline{I} = (x^2, y^2) \subset k[x, y, z]$, an ideal that is not 2-regular, and that is the ideal of a zero-dimensional intersection of X with a 2-plane. This concludes the proof of the corollary. ∎

Corollary 2.7 *The condition that a monomial ideal in $S = k[x_0, \ldots, x_r]$ satisfies property $\mathbf{N}_{2,p}$ for some $p \geq 1$, and in particular 2-regularity, is independent of the field k (not necessarily algebraically closed).* ∎

Remark 2.8 By a result of Bayer and Stillman [1987] (or see Eisenbud [1995] Theorem 15.20) a subscheme $X \subset \mathbb{P}^r$ over a field of characteristic zero is 2-regular if and only if it has a Borel-fixed (generic) initial ideal generated by quadratic monomials. Any scheme defined by a monomial ideal is, moreover, the degeneration by a flat family of linear sections, of a reduced union Y of planes defined by the monomials of a "polarization" (see for example Eisenbud [2004]). Thus each 2-regular scheme X over a field of characteristic zero, reduced or not, is associated canonically with an absolutely reduced scheme Y, a union of coordinate planes, that is also 2-regular.

Example 2.9 If X is the union of two disjoint lines in $\mathbb{P}^3$ with ideal $(a,b)\cap(c,d)\subset k[a,b,c,d]$ then X has generic initial ideal

$$(x^2, xy, y^2, xz) \subset k[x,y,z,w] = k[a,b,c,d],$$

and this ideal has polarization

$$(x_1x_2, x_1y_1, y_1y_2, x_1z) \subset k[x_1,x_2,y_1,y_2,z,w].$$

In this case the polarization scheme $Y \subset \mathbb{P}^5$ is the cone over the reduced union of two planes in $\mathbb{P}^3$ meeting a line "sticking out into" $\mathbb{P}^4$ in a point on the intersection of the two planes. Thus even if the original scheme is a union of planes, the resulting polarization of its generic initial ideal may be quite different. In this case the general hyperplane section of Y is the cone over the scheme consisting of 2 lines in $\mathbb{P}^2$ with a spatial embedded point of multiplicity one at their intersection – the limit of the original scheme X in a family where the two lines become coplanar.

3 Upper bounds for property $\mathbf{N}_p$

From an alternative perspective the results in Section 1 provide geometric explanations for the failure of property $\mathbf{N}_p$ and thus allow one to test optimality of the results of Green, Ein-Lazarsfeld, and many others, mentioned in the introduction.

Perhaps the simplest example (handled by different methods in Ottaviani-Paoletti [2001]) is the necessity of the conditions in the following:

Conjecture 3.1 *Property $\mathbf{N}_p$ holds for the d-uple embedding of $\mathbb{P}^n$ if and only if either*

- $n = 1$ *and* $d, p \in \mathbb{N}$, *or*
- $n = 2$, $d = 2$ *and* $p \in \mathbb{N}$, *or*
- $n \geq 3$, $d = 2$, *and* $p \leq 5$, *or*
- $n \geq 2$, $d \geq 3$ *and* $p \leq 3d - 3$.

Jozefiak-Pragacz-Weyman [1981] show that the 2-uple embedding of $\mathbb{P}^n$, $n \geq 3$, satisfies property $\mathbf{N}_5$. In the case of the d-uple embedding of $\mathbb{P}^2$ its minimal free resolution restricts to the minimal free resolution of a hyperplane section (a plane curve), and so Green [1984] implies that for $d \geq 3$ the d-uple embedding of $\mathbb{P}^2$ satisfies property $\mathbf{N}_{3d-3}$. See also Rubei [2003] for a proof of the fact that 3-uple embedding of $\mathbb{P}^n$ satisfies property $\mathbf{N}_4$ for all n. In all other cases the sufficiency of the conditions in Conjecture 3.1 is wide open. On the other hand Theorem 1.1 yields easily the necessity of those conditions, namely

Proposition 3.2 *Let $n \geq 2$ and $d \geq 2$ be integers.*
(a) *If $n \geq 2$ and $d \geq 3$, then the d-uple embedding of $\mathbb{P}^n$ fails property $\mathbf{N}_{3d-2}$.*
(b) *If $n \geq 3$, then the 2-uple embedding of $\mathbb{P}^n$ fails property $\mathbf{N}_6$.*

Proof. Observe first that for all $m < n$ the d-uple embedding of $\mathbb{P}^m$ is a linear section of the d-uple embedding of $\mathbb{P}^n$ and thus by Theorem 1.1 for the failure of property $\mathbf{N}_p$ it is enough to produce a $(p+2)$-secant p-plane to the d-uple embedding of $\mathbb{P}^m$ for some $m < n$.

To prove a) we may assume that $n = 2$. Since $d \geq 3$, a complete intersection $(3, d)$ in $\mathbb{P}^2$ is cut out by forms of degree d but fails to impose independent conditions on such forms. In other words, the linear span of the d-uple embedding of such a complete intersection is a $3d$-secant linear space of dimension $3d - 2$ to the d-uple embedding of $\mathbb{P}^2$, which therefore must fail property $\mathbf{N}_{3d-2}$ by Theorem 1.1.

Similarly, a complete intersection $Z \subset \mathbb{P}^3$ of three quadrics fails to impose independent conditions on quadrics, and thus maps via the 2-uple embedding of $\mathbb{P}^3$ to a collection of 8 points spanning only a six dimensional linear subspace Λ of $\mathbb{P}^9$ meeting the 2-uple embedding of $\mathbb{P}^3$ only along the points of Z. By Theorem 1.1, the 2-uple Veronese embedding of $\mathbb{P}^3$ fails property $\mathbf{N}_6$. ∎

The failure of property $\mathbf{N}_{3d-2}$ for the d-uple embedding of $\mathbb{P}^2$ can be accounted for also by the existence of a relatively long strand of linear syzygies in the minimal free resolution of $\omega_{\mathbb{P}^2}(d)$. Namely, with notations as in Eisenbud-Popescu-Schreyer-Walter [2002], we have the following

Proposition 3.3 *Let $W = H^0(\omega_{\mathbb{P}^2}(d))$ and set $w = \dim(W)$, let $U = H^0(\omega_{\mathbb{P}^2}^{-1})$, let $V = H^0(\mathcal{O}_{\mathbb{P}^2}(d))$ and $S = \mathrm{Sym}(V)$. If $d \geq 3$, the natural multiplication pairing $\mu : W \otimes U \to V$ makes $Q = \oplus_l(\wedge^{l+1}(W^*) \otimes \mathrm{Sym}^l(U^*))$ into a graded $E = \wedge^*(V^*)$-module such that the maximal irredundant quotient of the linear complex*

$$\mathbf{L}(Q^*):\ 0 \to \wedge^w W \otimes D_{w-1}(U) \otimes S(-w+1) \to \cdots \to \wedge^2 W \otimes U \otimes S(-1) \to W \otimes S$$

is a linear complex of the same length which injects as a degreewise direct summand into the minimal free resolution of the S-module $\oplus_{m \geq 0} H^0(\omega_{\mathbb{P}^2}(d(m+1)))$. In particular the d-uple embedding of $\mathbb{P}^2$ fails property $\mathbf{N}_{3d-2}$.

Proof. The above multiplication pairing μ is obviously geometrically 1-generic so the first part of the claim is a direct application of Proposition 2.10 in Eisenbud-Popescu-Schreyer-Walter [2002] with $L = \mathcal{O}_{\mathbb{P}^2}(d-3)$, $L' = \mathcal{O}_{\mathbb{P}^2}(3)$, and $L'' = \mathcal{O}_{\mathbb{P}^2}(d)$, and with W, U, and V as in the statement of the proposition.

For the second claim observe first that the homogeneous ideal I_d of the d-uple embedding of $\mathbb{P}^2$ is generated by quadrics but is only 3-regular so that its minimal free resolution has two strands (linear and quadratic). On the other hand the dual of the maximal irredundant quotient of the linear complex $\mathbf{L}(Q^*)$ has length $\binom{d-1}{2}$ and is a degreewise direct summand of the second strand into the minimal free resolution of I_d. Since the whole resolution of I_d has length $\binom{d+2}{2} - 3$ it follows that the d-uple embedding of $\mathbb{P}^2$ fails property $\mathbf{N}_{3d-2}$. ∎

The argument used in the proof of Proposition 3.2 a) provides upper bounds for property $\mathbf{N}_p$ for other Fano-type varieties and embeddings. For instance for embeddings of ruled or Del Pezzo surfaces we obtain the following bounds (where Proposition 3.2 a) is the case when $S = \mathbb{P}^2$):

Proposition 3.4 *Let S be a smooth surface and L be a very ample line bundle on S. If $|-K_S| \neq \emptyset$, and $\mathcal{O}(K_S) \otimes L$ is globally generated, then the image of S via the linear system $|L|$ fails property $\mathbf{N}_{-K_S \cdot L - 2}$.*

Proof. Let $D \in |-K_S|$, let $C \in |L|$ be a general curve and denote by $Z = D \cap C$ their intersection. The Koszul complex on the sections defining D and C expands to the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_C(-D) & \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{O}_Z & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_S(-D) & \longrightarrow & \mathcal{O}_S & \longrightarrow & \mathcal{O}_D & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_S(-D-C) & \longrightarrow & \mathcal{O}_S(-C) & \longrightarrow & \mathcal{O}_D(-C) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

which we twist by L and take cohomology. From the long exact sequence of the middle row, since $H^1(\mathcal{O}(-D) \otimes L) = H^1(\mathcal{O}(K_S) \otimes L) = 0$ by Kodaira vanishing (in characteristic 0) or by Shepherd-Barron [1991] and Terakawa [1999, Theorem 1.6] (in positive characteristic), we deduce that the natural restriction map $H^0(L) \to H^0(L_{|D})$ is surjective. On the other hand in the long exact sequence of the last column

$$\cdots \longrightarrow H^0(L_{|D}) \longrightarrow H^0(L_{|Z}) \longrightarrow H^1(\mathcal{O}_D(-C) \otimes L) \longrightarrow H^1(L_{|D}) \longrightarrow \cdots$$

we have $h^1(\mathcal{O}_D(-C) \otimes L) = h^0(\mathcal{O}_D) \geq 1$, while $h^1(L_{|D}) = h^0(L^{-1}_{|D}) = 0$ since $\mathcal{O}(K_D) = \mathcal{O}_D$ and L is ample. Putting everything together it follows that the subscheme Z fails to impose independent conditions on the sections of L. Moreover since $\mathcal{O}(K_S) \otimes L$ is globally generated we deduce that $\mathcal{I}_Z \otimes L$ is also globally generated. But length$(Z) = -K_S \cdot L$ so the claim of the proposition follows now directly from Theorem 1.1. ∎

Remark 3.5 1) By adjunction (see Sommese [1979] or Sommese-Van de Ven [1987]), in Proposition 3.4, the line bundle $\mathcal{O}(K_S) \otimes L$ is globally generated if and only if (S, L) is not one of the following pairs: $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, or $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ with E a rank 2 vector bundle on a curve.

2) A similar argument as in Proposition 3.4 shows that if X is a smooth projective surface and L is a very ample divisor on it, then the embedding of X via the linear system $|K_X + (p+3)L|$ fails to satisfy property $\mathbf{N}_{3pL^2-2}$, for $p \geq 3$ (or fails to satisfy property $\mathbf{N}_{(2p+2)L^2-2}$ for $p \geq 2$, if $(X, \mathcal{O}(L)) \neq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$).

Proposition 3.6 *Let X denote the image of the Segre-Veronese embedding*

$$\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_m} \xhookrightarrow{(d_1,d_2,\ldots,d_m)} \mathbb{P}^{\prod_{i=1}^{m} \binom{n_i+d_i}{d_i} - 1}$$

(*a*) *If $m \geq 3$ and $d_i = 1$ for at least three values of $1 \leq i \leq m$, then X fails property $\mathbf{N}_4$.*

(*b*) *If $m \geq 3$ and $d_i = 1$ for exactly two values of $1 \leq i \leq m$, then X fails property $\mathbf{N}_{2\min_{\{i|d_i>1\}} d_i + 2}$.*

(*c*) *If $m \geq 3$ and $d_i = 1$ for at most one value of $1 \leq i \leq m$, or if $m \geq 2$ and $d_i > 1$ for all $1 \leq i \leq m$, then X fails property $\mathbf{N}_{2\min_{\{i\neq j|d_i,d_j>1\}}(d_i+d_j)-2}$.*

Proof. We may argue as in the proof of Proposition 3.2 and exhibit for suitable p a p-dimensional linear subspace which is $(p+2)$-secant to the Segre-Veronese embedding of a product of $r < m$ factors. Failure of property $\mathbf{N}_p$ then follows from Theorem 1.1.

To prove a) we may assume that $m = 3$. The linear span of the Segre-Veronese embedding of a complete intersection of type $(1,1,1)^3$ is a 6-secant $\mathbb{P}^4$, thus X fails property $\mathbf{N}_4$ in this case.

Case b) is similar to a): we may assume that $m = 3$ and consider the linear span of the Segre-Veronese embedding of a complete intersection of one hypersurface of multidegree $(1,1,2)$, and two hypersurfaces of multidegree $(1,1,d)$ with $d = \min_{\{i|d_i>1\}} d_i$.

Finally in case c) we may assume that $m = 2$ and that both degrees are ≥ 2, in which case the claim follows from Proposition 3.4 for $S = \mathbb{P}^1 \times \mathbb{P}^1$. ∎

Not much is known concerning the converse of Proposition 3.6, except for computational evidence via Macaulay2 for small values of m, n_i and d_i. The following remark collects all positive related results we are aware of.

Remark 3.7 1) If $d_1, d_2 \geq 2$, then the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ via the linear system $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(d_1, d_2)|$ satisfies $\mathbf{N}_{2d_1+2d_2-3}$ (see Gallego-Purnaprajna [2001]), but fails to satisfy $\mathbf{N}_{2d_1+2d_2-2}$ by Proposition 3.6 or Proposition 3.4 above.

2) Lascoux [1978] and Pragacz-Weyman [1985] describe the minimal free resolution of the Segre embedding of $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$. In particular they show that it satisfies property $\mathbf{N}_p$ if and only if $p \leq 3$.

3) Using simplicial methods Rubei [2002, 2004] shows that the Segre embedding of $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_m}$ (at least three factors) satisfies property $\mathbf{N}_p$ if and only if $p \leq 3$. Also Corollary 8 in Rubei [2002] proves part b) in Proposition 3.6 via a

different method.

4) The resolution of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as well as a number of other special cases where the resolution is self-dual are investigated in Barcanescu-Manolache [1981].

Proposition 3.8 *The Plücker embedding of the Grassmannian* $\mathrm{Gr}(k,n) \subset \mathbb{P}^{\binom{n}{k}-1}$, *where* $2 \leq k \leq n-2$ *and* $n \geq 5$, *fails property* $\mathbf{N}_3$.

Proof. It is enough to observe that for all $2 \leq k \leq n-2$ and $n \geq 5$, the Plücker embedding of the Grassmannian $\mathrm{Gr}(k,n) \subset \mathbb{P}^{\binom{n}{k}-1}$ has as linear section the Plücker embedding of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$. On the other hand a general codimension three linear section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ is a collection of 5 points spanning only a $\mathbb{P}^3$, and the conclusion follows now again from Theorem 1.1. ∎

Remark 3.9 1) Jerzy Weyman informed us that property $\mathbf{N}_2$ always holds for the Plücker embedding of any Grassmannian.

2) Manivel [1996] proved that if $X = G/P$, where $G = \mathrm{SL}(V)$, V is a complex vector space and P a parabolic subgroup, and L is a very ample line bundle on X, then the embedding defined by the complete linear system $|L^p|$ satisfies property $\mathbf{N}_p$ for all $p \geq 1$.

Recall that a complete linear system $|L|$ on a projective variety X is said to be k-very ample if for any zero dimensional subscheme $Z \subset X$ of length $k+1$ the restriction map

$$H^0(L) \longrightarrow H^0(L_{|Z})$$

is surjective. In particular 0-very ample is "base point free" and 1-very ample is "very ample".

Pareschi [2000] and Pareschi-Popa [2003] proved that if X is an abelian variety and $L_1, \ldots, L_{p+3}$ are ample line bundles on X then the embedding of $X \subset \mathbb{P}^N$ by the linear system $|L_1 \otimes \ldots \otimes L_{p+3}|$ satisfies property $\mathbf{N}_p$. By Theorem 1.1 and the classification of small algebraic sets in Eisenbud-Green-Hulek-Popescu [2004] every positive dimensional reduced irreducible component of a linear section $\Lambda \cap X$, where Λ a linear subspace of dimension $\leq p$ of $\mathbb{P}^N$, is a variety of minimal degree in its linear span and hence rational. But abelian varieties do not contain rational positive dimensional subvarieties. Thus by Theorem 1.1 it follows that $L_1 \otimes \ldots \otimes L_{p+3}$ is $(p+1)$-very ample, which is a special case of Theorem 1 in Bauer-Szemberg [1997]).

Observe also that if $X = \prod_{i=1}^{\dim(X)} E_i$ is a product of elliptic curves, each with origin o_{E_i}, and $L := \prod_i p_i^*(\mathcal{O}_{E_i}(o_{E_i}))$ is the canonical principal polarization on X, then L^{p+3} fails to satisfy property $\mathbf{N}_{p+1}$. This is a consequence of Theorem 1.1 and Abel's theorem since one may choose $(p+3)$ points on E_i such that any divisor in the linear system $|(p+3)o_{E_i}|$ containing $(p+2)$ of those points contains also the remaining point.

Gross-Popescu [1998] conjectured that the general $(1,d)$-polarized abelian surface, for $d \geq 10$, satisfies property $\mathbf{N}_{[\frac{d}{2}]-4}$. As above, by Theorem 1.1, this would imply that a $(1,d)$-polarization on a general abelian surface is k-very ample if $d \geq 2k+3$ and $d \geq 10$ (compare again with Bauer-Szemberg [1997], Theorem 1).

4 Secants and syzygy varieties

In this section we analyze the restriction of linear syzygies to non-linear varieties with known syzygies such as rational normal curves, rational scrolls and Veronese surfaces.

Theorem 4.1 *Let $X, \Gamma \subset \mathbb{P}^r$ be subschemes such that X is non-degenerate and Γ is reduced with every irreducible component spanning all of $\mathbb{P}^r$. If the natural restriction map*

$$\mathrm{Tor}_p^S(I_{X\cup\Gamma}, k)_{p+2} \longrightarrow \mathrm{Tor}_p^S(I_X, k)_{p+2}$$

is not surjective, then

$$\dim \frac{H^0(\mathcal{I}_{X\cap\Gamma}(2))}{H^0(\mathcal{I}_\Gamma(2))} > p.$$

Proof. The main idea will be to use the Eisenbud-Koh-Stillman Conjecture **EKS** (proved by M. Green [1999]) which says that if $M = \oplus_{i\geq 0} M_i$ is a finitely generated graded module over the polynomial ring $S = \mathrm{Sym}(V)$ such that

a) $\ker(\wedge^p V \otimes M_0 \longrightarrow \wedge^{p-1} V \otimes M_1) \neq 0$, for some $p > 0$, and moreover

b) $\dim M_0 \leq p$,

then there exist a p dimensional family of rank one relations (i.e. decomposable tensors) in the kernel of the multiplication map $V \otimes M_0 \longrightarrow M_1$.

We will not need the full strength of **EKS**, but just the existence of such rank one relations under the above hypothesis, and we will apply **EKS** to

$$M = \bigoplus_{i\geq 0} \frac{H^0(\mathcal{I}_{X\cap\Gamma}(i+2))}{H^0(\mathcal{I}_\Gamma(i+2))}$$

regarded as a finitely generated module over $S = \mathrm{Sym}(V)$, the polynomial ring of the ambient $\mathbb{P}^r$.

There are no rank one relations in the kernel of the multiplication morphism $V \otimes M_0 \longrightarrow M_1$. Such a rank one relation would amount to the existence of a quadric defined by $Q \in H^0(\mathcal{I}_{X\cap\Gamma}(2))$ not vanishing on Γ and a hyperplane defined by $H \in H^0(\mathcal{O}_{\mathbb{P}^r}(1))$ such that $QH \in H^0(\mathcal{I}_\Gamma(3))$, which is impossible since each irreducible component of Γ is assumed to be nondegenerate.

We will relate condition *a*) in **EKS** for the module M to the analogous one for the module

$$P = \bigoplus_{i\geq 0} \frac{H^0(\mathcal{I}_X(i+2))}{H^0(\mathcal{I}_{X\cup\Gamma}(i+2))}.$$

Expressing as usual the Tor's via Koszul cohomology our hypothesis that

$$\mathrm{Tor}_p^S(I_{X\cup\Gamma}, k)_{p+2} \longrightarrow \mathrm{Tor}_p^S(I_X, k)_{p+2}$$

is not surjective translates into the existence of an element

$$\alpha \in \ker(\wedge^p V \otimes H^0(\mathcal{I}_X(2)) \longrightarrow \wedge^{p-1} V \otimes H^0(\mathcal{I}_X(3)))$$

which is not in the image of the natural inclusion morphism

$$\wedge^p V \otimes H^0(\mathcal{I}_{X\cup\Gamma}(2)) \longrightarrow \wedge^p V \otimes H^0(\mathcal{I}_X(2)).$$

Taking global sections in the first row of the exact diagram of ideal sheaves

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}_{X\cup\Gamma}(2) \arrow[r] \arrow[d, hook] & \mathcal{I}_X(2) \arrow[r] \arrow[d, hook] & \mathcal{I}_{X\cap\Gamma,\Gamma}(2) \arrow[d, "\cong"'] & \\ 0 \arrow[r] & \mathcal{I}_\Gamma(2) \arrow[r] & \mathcal{I}_{X\cap\Gamma}(2) \arrow[r] & \mathcal{I}_{X\cap\Gamma,\Gamma}(2) \arrow[r] & 0 \end{tikzcd}$$

we see that α induces a non-trivial element $\bar{\alpha}$ in

$$\bar{\alpha} \in \ker(\wedge^p V \otimes P_0 \longrightarrow \wedge^{p-1} V \otimes P_1).$$

On the other hand, twisting and taking global sections in the above diagram yields the inclusion $P \subseteq M$. In particular, we may view $\bar{\alpha}$ as an element of $\ker(\wedge^p V \otimes M_0 \longrightarrow \wedge^{p-1} V \otimes M_1)$, which is thus non-zero. By **EKS**, since there are no rank one relations in the kernel of $V \otimes M_0 \longrightarrow M_1$, we deduce that $\dim M_0 > p$ which finishes the proof of the theorem. ∎

In the case where Γ is a smooth curve Theorem 4.1 has more geometric content and so we restate a special case of it explicitly:

Corollary 4.2 *With notation as in Theorem 4.1, if Γ is an irreducible nondegenerate curve such that*

$$\mathrm{Tor}_p^S(I_{X\cup\Gamma}, k)_{p+2} \longrightarrow \mathrm{Tor}_p^S(I_X, k)_{p+2}$$

is not surjective, then $h^0(\mathcal{O}_\Gamma(2H - X\cap\Gamma)) > p$. In particular, if X satisfies property $\mathbf{N}_{2,p}$ and Γ is a rational normal curve in $\mathbb{P}^r$ not contained in X, then

$$\mathrm{length}(X\cap\Gamma) < 2r+1-p.$$

Proof. The first part follows from the conclusion of Theorem 4.1 and the cohomology of a twist of the short exact sequence

$$0 \longrightarrow \mathcal{I}_\Gamma \longrightarrow \mathcal{I}_{\Gamma\cap X} \longrightarrow \mathcal{O}_\Gamma(-X\cap\Gamma) \longrightarrow 0.$$

For the second part observe that the restriction map

$$\mathrm{Tor}_p^S(I_{X\cup\Gamma}, k)_{p+2} \longrightarrow \mathrm{Tor}_p^S(I_X, k)_{p+2}$$

is not surjective since every element of $\mathrm{Tor}_p^S(I_{X\cup\Gamma}, k)_{p+2}$ is represented by a syzygy among the quadrics in $I_{X\cup\Gamma}$, which are a subset of those in I_Γ. But a non-trivial syzygy among the quadrics of Γ involves all quadrics containing Γ and thus its syzygy variety is all of Γ (see for instance Ehbauer [1994] or Eisenbud-Popescu [1999]). Thus the map of Tor's is not surjective if X is not contained in Γ. We may conclude now by Theorem 4.1 since $h^0(\mathcal{O}_\Gamma(2H - X\cap\Gamma)) = h^0(\mathcal{O}_{\mathbb{P}^1}(2r - \mathrm{length}(X\cap\Gamma))) > p$ if and only if $\mathrm{length}(X\cap\Gamma) < 2r+1-p$. ∎

Remark 4.3 In the special case where both X and Γ are rational normal curves in $\mathbb{P}^r$, Corollary 4.2 yields that X and Γ can meet at most in $2r+1-(r-1) = r+2$ points. In case of equality, the union of the two rational normal curves is a degeneration of a canonical curve in $\mathbb{P}^r$ (a so called "binary" curve). See also Eisenbud-Harris [1992], or Diaz [1986] and Giuffrida [1988] for related results.

Remark 4.4 The second part of Corollary 4.2 fails if, for instance, Γ does not span all of $\mathbb{P}^r$. Here is an easy counterexample for $p = 2$: Let X be the cone over a rational normal curve in $\mathbb{P}^4$, say

$$X = \left\{x \mid \operatorname{rank}\begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix} \le 1\right\} \subset \mathbb{P}^5 = \mathbb{P}^5(x_0, x_1, x_2, x_3, x_4, x_5)$$

and let $\Lambda = \{x_2 = x_3 = x_5 = 0\} \subset \mathbb{P}^5$. Then $X \cap \Lambda = \{x_2 = x_3 = x_5 = x_1x_4 = 0\}$ which is a degenerate conic (union of two lines). Now if Γ is any smooth conic in Λ, then $\operatorname{length}(X \cap \Gamma) = 4$, whereas Corollary 4.2 gives an upper bound $2\cdot 2+1-2 = 3$.

Theorem 4.1 has numerous applications. We list in the sequel the most interesting ones. The first one is Green's syzygetic Castelnuovo lemma (see also Ehbauer [1994], Yanagawa [1994], and Eisenbud-Popescu [1999]):

Corollary 4.5 *Let $X \subset \mathbb{P}^r$ be a finite subscheme which contains a subscheme of length $r+3$ in linearly general position. If $\operatorname{Tor}_{r-2}(I_X, k)_r \neq 0$, then X lies on a (unique) smooth rational normal curve.*

Proof. Let $X' \subset X$ be a subscheme of length $r+3$ in linearly general position, and let $\Gamma \subset \mathbb{P}^r$ be the unique rational normal curve containing X' (see Eisenbud-Harris [1992], or Eisenbud-Popescu [2000]). Suppose that X is not contained in Γ. Then the hypotheses of Theorem 4.1 are satisfied for the scheme X and the rational normal curve Γ. More precisely, as in the proof of Corollary 4.2, the restriction map

$$\operatorname{Tor}^S_{r-2}(I_{X\cup\Gamma}, k)_r \longrightarrow \operatorname{Tor}^S_{r-2}(I_X, k)_r$$

is not surjective if X is not contained in Γ. We deduce from Theorem 4.1 that $h^0(\mathcal{O}_\Gamma(2H - (X \cap \Gamma))) > r-2$. This translates into $h^0(\mathcal{O}_{\mathbb{P}^1}(2r - \operatorname{length}(X \cap \Gamma))) > r-2$, whence $\operatorname{length}(X \cap \Gamma) \le r+2$, a contradiction since $X \cap \Gamma$ already contains X' of length $r+3$. It follows that X is contained in Γ and this concludes the proof of the corollary. ∎

Similarly the above techniques yield the following amusing fact (see Eisenbud-Hulek-Popescu [2003] for more details and a better bound):

Proposition 4.6 *Two Veronese surfaces in $\mathbb{P}^5$ whose intersection is zero-dimensional meet in a scheme of degree at most* 12. ∎

Coble [1922] and Conner [1911] show that it is possible to realize Veronese surfaces meeting in 10 points, and describe such collections of points in terms of association. A detailed analysis of the possibilities of how two Veronese surfaces can intersect in a scheme of finite length can be found in our paper Eisenbud-Hulek-Popescu [2003]. We show that the length of such a scheme is at most 10. Moreover, we prove that in the case of transversal intersection two Veronese surfaces meet in either 10 points (in which case they lie on a common quadric) or in at most 8 points. We give there also a modern account of some of Coble and Conner's results.

Similar results hold for zero-dimensional intersections of scrolls:

Proposition 4.7 *Let X and Γ be two nondegenerate scrolls of dimensions m and n, respectively, in $\mathbb{P}^r$ with $m \le n$ and such that $X \cap \Gamma$ is a zero dimensional scheme. Then* $\operatorname{length}(X \cap \Gamma) \le nr + m - \binom{n}{2} + 1$.

Proof. One applies Theorem 4.1 as in Corollary 4.2. The knowledge of the number of independent quadrics in the ideal of a scroll (use for instance the Eagon-Northcott complex) gives then the claimed bound via direct computation. ∎

References

A.B. Coble: Associated sets of points, *Trans. Amer. Math. Soc.* **24** (1922), no.1, 1–20.

S. Barcanescu, N. Manolache: Betti numbers of Segre-Veronese singularities, *Rev. Roumaine Math. Pures Appl.* **26** (1981), no. 4, 549–565.

Th. Bauer, T. Szemberg: Higher order embeddings of abelian varieties, *Math. Ann.* **224** (1997), no. 3, 449-455.

Th. Bauer, T. Szemberg: Primitive higher order embeddings of abelian surfaces, *Trans. Amer. Math. Soc.* **349** (1997), no. 4, 1675–1683.

D. Bayer, M. Stillman: A criterion for detecting m-regularity. *Invent. Math.* **87**, (1987), no. 1, 1–11.

G. Caviglia: Bounds on the Castelnuovo-Mumford regularity of tensor products, preprint 2003.

A.B. Coble: Associated sets of points, *Trans. Amer. Math. Soc.* **24** (1922), 1–20.

J.R. Conner: Basic systems of rational norm-curves, *Amer J. Math.* **32** (1911), no. 2, 115–176.

S. Diaz: Space curves that intersect often, *Pacific J. Math.* **123** (1986), no. 2, 263–267.

I. Dolgachev, D. Ortland: Points sets in projective spaces and theta functions, *Astérisque* **165**, (1988).

S. Ehbauer: Syzygies of points in projective space and applications, in *"Zero-dimensional schemes (Ravello, 1992)"*, 145–170, de Gruyter, Berlin, 1994.

L. Ein, R. Lazarsfeld: Syzygies and Koszul cohomology of smooth projective varieties of arbitrary dimension, *Invent. Math.* **111** (1993), no. 1, 51–67.

D. Eisenbud: *Commutative Algebra with a View Toward Algebraic Geometry*, Springer, New York, 1995.

D. Eisenbud: *Geometry of Syzygies*, Springer, New York, forthcoming book, 2004.

D. Eisenbud, S. Goto: Linear free resolutions and minimal multiplicity. *J. Algebra*, **88** (1984), no. 1, 89–133.

D. Eisenbud, M. Green, K. Hulek, S. Popescu: The geometry of 2-regular algebraic sets, preprint 2004.

D. Eisenbud, J. Harris: On varieties of minimal degree (a centennial account). Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 3–13, *Proc. Sympos. Pure Math.*, **46**, Part 1.

D. Eisenbud, J. Harris: Finite projective schemes in linearly general position. *J. Algebraic Geom.* **1** (1992), no. 1, 15–30.

D. Eisenbud, J. Harris: An intersection bound for rank 1 loci, with applications to Castelnuovo and Clifford theory. *J. Algebraic Geom.* **1** (1992), no. 1, 31–59.

D. Eisenbud, K. Hulek, S. Popescu: A note on the intersection of Veronese surfaces, in *"Proceedings of the NATO Advanced Workshop, held in Sinaia, Romania 2002"*, J. Herzog, V. Vuletescu eds, NATO Science series, Kluwer, 2003, 127–139.

D. Eisenbud, K. Huneke, B. Ulrich: Linearly presented ideals and a subadditivity formula for the degrees of syzygies, preprint 2004.

D. Eisenbud, J. Koh: Some linear syzygy conjectures. *Adv. Math.* **90** (1991), no. 1, 47–76.

D. Eisenbud, S. Popescu: Syzygy ideals for determinantal ideals and the syzygetic Castelnuovo lemma. In *"Commutative Algebra, Algebraic Geometry, and Computational Methods, Hanoi 1996"*, 247–258, Springer Verlag 1999

D. Eisenbud, S. Popescu: The projective geometry of the Gale Transform, *J. Algebra* **230** (2000), no. 1, 127–173.

D. Eisenbud, S. Popescu, F.-O. Schreyer, Ch. Walter: Exterior algebra methods for the minimal resolution conjecture, *Duke Math. J.* **112** (2002), no. 2, 379–395.

R. Fröberg: On Stanley-Reisner rings. In *"Topics in algebra, Part 2" (Warsaw, 1988)*, Banach Center Publ., **26**, Part 2, 57–70, PWN, Warsaw, 1990.

F. Gallego, B. Purnaprajna: Some results on rational surfaces and Fano varieties, *J. Reine Angew. Math.* **538** (2001), 25–55.

S. Giuffrida: On the intersection of two curves in $\mathbb{P}^r$, *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.* **122** (1988), no. 3-4, 139–143.

D. Grayson, M. Stillman: *Macaulay2*: A computer program designed to support computations in algebraic geometry and computer algebra. Source and object code available from `http://www.math.uiuc.edu/Macaulay2/`.

M. Green: Koszul cohomology and the geometry of projective varieties, *J. Differential Geom.* **19** (1984), no. 1, 125–171.

M. Green: The Eisenbud-Koh-Stillman conjecture on linear syzygies, *Invent. Math.* **136** (1999), no. 2, 411–418.

M. Green, R. Lazarsfeld: On the projective normality of complete linear series on an algebraic curve. *Invent. Math.* **83** (1985), no. 1, 73–90.

M. Green, R. Lazarsfeld: Some results on the syzygies of finite sets and algebraic curves, *Compositio Math.* **67** (1988), no. 3, 301–314.

M. Gross, S. Popescu: Equations of $(1, d)$-polarized abelian surfaces, *Math. Ann.* **310** (1998), no. 2, 333–377.

J. Herzog, T. Hibi, X. Zheng: Dirac's theorem on chordal graphs and Alexander duality, preprint 2003.

M. Hochster: Cohen-Macaulay rings, combinatorics and simplicial complexes, in "Ring theory II", McDonald B.R., Morris, R. A. (eds), *Lecture Notes in Pure and Appl. Math.*, **26**, M. Dekker 1977.

T. Jozefiak, P. Pragacz, J. Weyman: Resolutions of determinantal varieties and tensor complexes associated with symmetric and antisymmetric matrices, *Asterisque* **87-88** (1981), 109–189.

A. Lascoux: Syzygies des variétés determinantales, *Adv. in Math.* **30** (1978), no. 3, 202–237.

R. Lazarsfeld: A sampling of vector bundle techniques in the study of linear series. In *"Lectures on Riemann surfaces"* (Trieste, 1987), 500–559, World Sci. Publishing, Teaneck, NJ, 1989.

L. Manivel: On the syzygies of flag manifolds, *Trans. Amer. Math. Soc.* **124** (1996), no. 8, 2293–2299.

G. Ottaviani, R. Paoletti: Syzygies of Veronese embeddings, *Compositio Math.* **125**, no. 1, (2001), 31–37.

G. Pareschi: Syzygies of abelian varieties. *J. Amer. Math. Soc.* **13** (2000), no. 3, 651–664.

G. Pareschi, M. Popa: Regularity on abelian varieties I, *J. Amer. Math. Soc.* **16** (2003), no. 2, 285–302.

P. Pragacz, J. Weyman: Complexes associated with trace and evaluation. Another approach to Lascoux's resolution, *Adv. in Math.* **57** (1985), no. 2, 163–207.

E. Rubei: On syzygies of Segre embeddings, *Proc. Amer. Math. Soc.* **130** (2002), no. 12, 3483–3493.

E. Rubei: A result on resolutions of Veronese embeddings, preprint `math.AG/0309102`.

E. Rubei: Resolutions of Segre embeddings of projective spaces of any dimension, preprint `math.AG/0404417`.

N. Shepherd-Barron, Unstable vector bundles and linear systems on surfaces in characteristic p, *Invent. Math.* **106** (1991), no. 2, 243–262.

V. Shokurov: The Noether-Enriques theorem on canonical curves, *Mat. Sbornik* **15** (1971), 361–403 (engl. transl).

A.J. Sommese: Hyperplane sections of projective surfaces I. The adjunction mapping, *Duke Math. J.* **46** (1979), no. 2, 377–401.

A.J. Sommese, A. Van de Ven: On the adjunction mapping, *Math. Ann.* **278** (1987), no. 1-4, 593–603.

E. Spanier: *Algebraic Topology*, McGraw-Hill Book Co., 1966.

R. Stanley: *Combinatorics and Commutative Algebra*, Second edition, Progress in Math. **41**, Birkhäuser, 1996.

H. Terakawa: The d-very ampleness on a projective surface in positive characteristic, *Pacific J. of Math.* **187** (1999), no. 1, 187–198.

P. Vermeire: Some results on secant varieties leading to a geometric flip construction, *Compositio Math.* **125** (2001), no. 3, 263–282.

K. Yanagawa: Some generalizations of Castelnuovo's lemma on zero-dimensional schemes. *J. Alg.* **170** (1994), no. 2, 429–431.

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