Appendix

We collect in this chapter numerical data about the constructed or studied surfaces and give references to the points in the thesis where the corresponding additional information can be found.

1. Degree 10

10.A.

Invariants
$d = 10,\ \pi = 9,\ HK = 6,\ K^2 = -3,\ \chi = 2,\ q = 0,\ N_6 = 3$

Classification
$S = S_{min}(p_1, p_2, p_3)$ non-minimal $K3$ surface
$H = H_{min} - 4E_1 - E_2 - E_3$

Syzygies of $\mathcal{I}_S$

$$\begin{array}{cccccccc} & \mathcal{O}(-4) & & & & & & \\ & \oplus & & & & & & \\ 0 \leftarrow \mathcal{I}_S \leftarrow & 9\mathcal{O}(-5) & \nwarrow & 15\mathcal{O}(-6) & & 7\mathcal{O}(-7) & & \mathcal{O}(-8) \\ & \oplus & & \oplus & \longleftarrow & \oplus & \longleftarrow & \oplus \leftarrow 0 \\ & \mathcal{O}(-6) & & 3\mathcal{O}(-7) & & 3\mathcal{O}(-8) & & \mathcal{O}(-9) \end{array}$$

Construction
$\mathcal{E} = \mathcal{O}(-1) \oplus \Omega^3(3),\ \mathcal{F} = \ker(5\mathcal{O} \oplus 2\mathcal{O}(1) \xrightarrow{\psi} \mathcal{O}(2))$
where ψ is a special morphism.
see (2.2) for details and (2.4) for a liaison construction,

Cohomology

$\uparrow$ i

1				
	1			
		1	3	1
				1

$\rightarrow$ p

$h^i(\mathcal{I}_S(p))$

Family properties
irreducible, unirational, of dimension 45
which differs from $\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 44$

2. Degree 11

11.A.

Invariants
$d = 11,\ \pi = 10,\ HK = 7,\ K^2 = -6,\ \chi = 1,\ q = 0,\ N_6 = 10$, [DES]

Classification
$S = S_{min}(p_1, ..., p_6)$ non-minimal Enriques surface
$H = H_{min} - 2E_1 - \sum_2^6 E_i$

Syzygies of $\mathcal{I}_S$

$$\begin{array}{cccccc} & 15\mathcal{O}(-5) & & 3\mathcal{O}(-6) & & \\ 0 \leftarrow \mathcal{I}_S \leftarrow & \oplus & \longleftarrow & \oplus & \nwarrow & \\ & 10\mathcal{O}(-6) & & 26\mathcal{O}(-7) & & 20\mathcal{O}(-8) \longleftarrow 5\mathcal{O}(-9) \leftarrow 0 \end{array}$$

Construction
$\mathcal{E} = 2\Omega^3(3),\ \mathcal{F} = \ker(10\mathcal{O} \xrightarrow{\psi} \mathcal{O}(2))$
where ψ is a special morphism.
see [DES] and (3.7) for details
see also (3.8) for a liaison construction

Cohomology

i $\uparrow$

	2			
		1	5	5

$\rightarrow$ p

$h^i(\mathcal{I}_S(p))$

Family properties
irreducible, unirational, of dimension 34
which differs from $\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 36$

11.B1.

Invariants
$d = 11,\ \pi = 11,\ HK = 9,\ K^2 = -11,\ \chi = 1,\ q = 0,\ N_6 = 7$, [DES]

Classification
$S = \mathbb{P}^2(p_0, ..., p_{19})$ rational surface
$H = 10l - 4E_0 - 3\sum_1^3 E_i - 2\sum_4^{13} E_j - \sum_{14}^{19} E_k$

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{matrix} 10\mathcal{O}(-5) \\ \oplus \\ \mathcal{O}(-6) \end{matrix} \longleftarrow \begin{matrix} 13\mathcal{O}(-6) \\ \oplus \\ 3\mathcal{O}(-7) \end{matrix} \longleftarrow \begin{matrix} 4\mathcal{O}(-7) \\ \oplus \\ 3\mathcal{O}(-8) \end{matrix} \nwarrow \mathcal{O}(-9) \leftarrow 0$$

Construction
$\mathcal{E} = 3\Omega^3(3)$, $\mathcal{F} = \Omega^2(2) \oplus \ker(9\mathcal{O} \xrightarrow{\psi} 2\mathcal{O}(1))$
where ψ is a general morphism.
see (3.23) for details

Cohomology

	3	1		
			2	1

$h^i(\mathcal{I}_S(p))$ (axes: i vertical, p horizontal)

Family properties
irreducible, unirational
$\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 41$

11.B2.

Invariants
$d = 11,\ \pi = 11,\ HK = 9,\ K^2 = -11,\ \chi = 1,\ q = 0,\ N_6 = 7$

Classification
$S = \mathbb{P}^2(p_0, ..., p_{19})$ rational surface
$H = 11l - 5E_0 - 3\sum_1^6 E_i - 2\sum_7^{12} E_j - \sum_{13}^{19} E_k$

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow 10\mathcal{O}(-5) \nwarrow \begin{matrix} 12\mathcal{O}(-6) \\ \oplus \\ 2\mathcal{O}(-7) \end{matrix} \longleftarrow \begin{matrix} 3\mathcal{O}(-7) \\ \oplus \\ 3\mathcal{O}(-8) \end{matrix} \nwarrow \mathcal{O}(-9) \leftarrow 0$$

Construction
$\mathcal{E} = 3\Omega^3(3)$, $\mathcal{F} = \ker(\Omega^2(2) \oplus 2\Omega^1(1) \xrightarrow{\psi} \mathcal{O})$
where ψ is a general morphism.
see (3.23) for details

Cohomology

	3	1		
			2	1

$h^i(\mathcal{I}_S(p))$ (axes: i vertical, p horizontal)

Family properties
irreducible, unirational
$\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 41$

11.B3.

Invariants
$d = 11,\ \pi = 11,\ HK = 9,\ K^2 = -11,\ \chi = 1,\ q = 0,\ N_6 = 7$

Classification
$S = \mathbb{P}^2(p_0, ..., p_{19})$ rational surface
$H = 13l - 5E_0 - 4\sum_1^7 E_i - 2\sum_8^{10} E_j - \sum_{11}^{19} E_k$

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{matrix} 10\mathcal{O}(-5) \\ \oplus \\ 2\mathcal{O}(-6) \end{matrix} \longleftarrow \begin{matrix} 14\mathcal{O}(-6) \\ \oplus \\ 5\mathcal{O}(-7) \end{matrix} \longleftarrow \begin{matrix} 6\mathcal{O}(-7) \\ \oplus \\ 4\mathcal{O}(-8) \end{matrix} \longleftarrow \begin{matrix} \mathcal{O}(-8) \\ \oplus \\ \mathcal{O}(-9) \end{matrix} \leftarrow 0$$

Construction
$\mathcal{E} = 3\Omega^3(3)$, $\mathcal{F} = \ker(\Omega^2(2) \oplus 2\Omega^1(1) \xrightarrow{\psi} \mathcal{O})$
where ψ is a special morphism.
see (3.23) for details

Cohomology

	3	1		
			2	1

$h^i(\mathcal{I}_S(p))$ (axes: i vertical, p horizontal)

Family properties
irreducible, unirational
$\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 41$

11.C1.

Invariants

$d = 11,\ \pi = 11,\ HK = 9,\ K^2 = -5,\ \chi = 2,\ q = 0,\ N_6 = 4$, [DES]

Classification

$S = S_{min}(p_1, ..., p_5)$ non-minimal $K3$ surface

$H = H_{min} - 5E_1 - \sum_2^5 E_i$

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow 9\mathcal{O}(-5) \nwarrow \begin{matrix} 8\mathcal{O}(-6) \\ \oplus \\ 5\mathcal{O}(-7) \end{matrix} \nwarrow 7\mathcal{O}(-8) \longleftarrow 2\mathcal{O}(-9) \leftarrow 0$$

Construction

$\mathcal{E} = \mathcal{O} \oplus 2\Omega^3(3)$, $\mathcal{F} = \ker(13\mathcal{O} \xrightarrow{\psi} 3\mathcal{O}(1))$

where ψ is a general morphism.

see [DES] or (3.30) for details

Cohomology

$\mathrm{h}^i(\mathcal{I}_S(p))$ (rows i, columns p)

1				
	2			
			3	2

Family properties

irreducible, unirational

$\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 43$

11.C2.

Invariants

$d = 11,\ \pi = 11,\ HK = 9,\ K^2 = -5,\ \chi = 2,\ q = 0,\ N_6 = 4$

Classification

$S = S_{min}(p_1, ..., p_5)$ non-minimal $K3$ surface

$H = H_{min} - 4E_1 - 2E_2 - \sum_3^5 E_i$

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{matrix} 9\mathcal{O}(-5) \\ \oplus \\ \mathcal{O}(-6) \end{matrix} \longleftarrow \begin{matrix} 9\mathcal{O}(-6) \\ \oplus \\ 6\mathcal{O}(-7) \end{matrix} \longleftarrow \begin{matrix} \mathcal{O}(-7) \\ \oplus \\ 7\mathcal{O}(-8) \end{matrix} \nwarrow 2\mathcal{O}(-9) \leftarrow 0$$

Construction

$\mathcal{E} = \mathcal{O} \oplus 2\Omega^3(3)$, $\mathcal{F} = \ker(13\mathcal{O} \xrightarrow{\psi} 3\mathcal{O}(1))$

where ψ is a special morphism.

see (3.30) for more details

Cohomology

$\mathrm{h}^i(\mathcal{I}_S(p))$ (rows i, columns p)

1				
	2			
			3	2

Family properties

irreducible, unirational

$\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 43$

11.C3.

Invariants

$d = 11,\ \pi = 11,\ HK = 9,\ K^2 = -5,\ \chi = 2,\ q = 0,\ N_6 = 4,$

Classification

$S = S_{min}(p_1, ..., p_5)$ non-minimal $K3$ surface

$H = H_{min} - 3E_1 - 2\sum_2^3 E_i - \sum_4^5 E_j$

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{matrix} 9\mathcal{O}(-5) \\ \oplus \\ 2\mathcal{O}(-6) \end{matrix} \longleftarrow \begin{matrix} 10\mathcal{O}(-6) \\ \oplus \\ 7\mathcal{O}(-7) \end{matrix} \longleftarrow \begin{matrix} 2\mathcal{O}(-7) \\ \oplus \\ 7\mathcal{O}(-8) \end{matrix} \nwarrow 2\mathcal{O}(-9) \leftarrow 0$$

Construction

$\mathcal{E} = \mathcal{O} \oplus 2\Omega^3(3)$, $\mathcal{F} = \ker(13\mathcal{O} \xrightarrow{\psi} 3\mathcal{O}(1))$

where ψ is a special morphism.

see (3.30) for more details

Cohomology

$\mathrm{h}^i(\mathcal{I}_S(p))$ (rows i, columns p)

1				
	2			
			3	2

Family properties

irreducible, unirational

$\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 43$

11.C4.

Invariants
$d = 11,\ \pi = 11,\ HK = 9,\ K^2 = -5,\ \chi = 2,\ q = 0,\ N_6 = 4,$

Cohomology

1				
	2			
			3	2

$h^i(\mathcal{I}_S(p))$

Classification
$S = S_{min}(p_1, ..., p_5)$ non-minimal $K3$ surface
$H = H_{min} - 2\sum_1^4 E_i - E_5$

Syzygies of $\mathcal{I}_S$

$$0\leftarrow \mathcal{I}_S \leftarrow \begin{array}{c}9\mathcal{O}(-5)\\ \oplus\\ 3\mathcal{O}(-6)\end{array} \longleftarrow \begin{array}{c}11\mathcal{O}(-6)\\ \oplus\\ 8\mathcal{O}(-7)\end{array} \longleftarrow \begin{array}{c}3\mathcal{O}(-7)\\ \oplus\\ 7\mathcal{O}(-8)\end{array} \nwarrow 2\mathcal{O}(-9) \leftarrow 0$$

Construction
$\mathcal{E} = \mathcal{O} \oplus 2\Omega^3(3)$, $\mathcal{F} = \ker(13\mathcal{O} \xrightarrow{\psi} 3\mathcal{O}(1))$
where ψ is a special morphism.
see (3.30) for more details

Family properties
irreducible, unirational
$\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 43$

11.D.

Invariants
$d = 11,\ \pi = 11,\ HK = 9,\ K^2 = 1,\ \chi = 3,\ q = 0,\ N_6 = 5,$

Cohomology

2				
	1			
		1	4	3

$h^i(\mathcal{I}_S(p))$

Classification
$S = S_{min}(p_1)$ non-minimal general type surface
$H = H_{min} - E_1$

Syzygies of $\mathcal{I}_S$

$$0\leftarrow \mathcal{I}_S \leftarrow \begin{array}{c}8\mathcal{O}(-5)\\ \oplus\\ 4\mathcal{O}(-6)\end{array} \longleftarrow \begin{array}{c}8\mathcal{O}(-6)\\ \oplus\\ 12\mathcal{O}(-7)\end{array} \longleftarrow \begin{array}{c}\mathcal{O}(-7)\\ \oplus\\ 11\mathcal{O}(-8)\end{array} \nwarrow 3\mathcal{O}(-9) \leftarrow 0$$

Construction
$\mathcal{E} = 2\mathcal{O}(-1) \oplus \Omega^3(3)$, $\mathcal{F} = \ker(7\mathcal{O} \oplus \mathcal{O}(1) \xrightarrow{\psi} \mathcal{O}(2))$
where ψ is a special morphism.
see (3.32) for more details

Family properties
irreducible, unirational
$\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 45$

11.E.

Invariants
$d = 11,\ \pi = 12,\ HK = 11,\ K^2 = -10,\ \chi = 2,\ q = 0,\ N_6 = 9$

Cohomology

1				
	3	2		
				2

$h^i(\mathcal{I}_S(p))$

Classification
$S = S_{min}(p_1, ..., p_{10})$ non-minimal $K3$ surface
$H = H_{min} - 2E_1 - \sum_2^{10} E_i$

Syzygies of $\mathcal{I}_S$

$$0\leftarrow \mathcal{I}_S \leftarrow \begin{array}{c}2\mathcal{O}(-4)\\ \oplus\\ 4\mathcal{O}(-5)\end{array} \nwarrow 7\mathcal{O}(-6) \longleftarrow 2\mathcal{O}(-7) \leftarrow 0$$

Construction
$\mathcal{E} = \mathcal{O}(-1) \oplus 2\Omega^3(3)$, $\mathcal{F} = 2\Omega^2(2) \oplus 2\mathcal{O}$
see (3.38) for more details
see (3.39) and (3.41) for liaison

Family properties
irreducible, unirational, of dimension 48
$\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 48$

11.F1.

Invariants
$d = 11,\ \pi = 12,\ HK = 11,\ K^2 = -4,\ \chi = 3,\ q = 0,\ N_6 = 3$

Classification
$S = S_{min}(p_0, ..., p_3)$ non-minimal proper elliptic surface
$H = H_{min} - 2E_0 - \sum_1^3 E_i$

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{array}{c} \mathcal{O}(-4) \\ \oplus \\ 8\mathcal{O}(-5) \end{array} \nwarrow 13\mathcal{O}(-6) \leftarrow 6\mathcal{O}(-7) \leftarrow \mathcal{O}(-8) \leftarrow 0$$

Construction
$\mathcal{E} = 2\mathcal{O}(-1) \oplus 2\Omega^3(3),\ \mathcal{F} = \Omega^2(2) \oplus \Omega^1(1) \oplus \mathcal{O}$
see (3.43) for more details
see also (3.44) for a liaison construction

Cohomology

$h^i(\mathcal{I}_S(p))$ (rows i, top to bottom; columns p, left to right)

2				
	2	1		
			1	
				1

Family properties
irreducible, unirational, of dimension 50
which equals $\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 50$

11.F2.

Invariants
$d = 11,\ \pi = 12,\ HK = 11,\ K^2 = -4,\ \chi = 3,\ q = 0,\ N_6 = 3$

Classification
$S = S_{min}(p_1, ..., p_5)$ blown-up, general type Horikawa surface
$H = H_{min} - \sum_1^5 E_i$

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{array}{c} 2\mathcal{O}(-4) \\ \oplus \\ 3\mathcal{O}(-5) \\ \oplus \\ 2\mathcal{O}(-6) \end{array} \nwarrow \begin{array}{c} 5\mathcal{O}(-6) \\ \oplus \\ 5\mathcal{O}(-7) \end{array} \longleftarrow \begin{array}{c} \mathcal{O}(-7) \\ \oplus \\ 4\mathcal{O}(-8) \end{array} \nwarrow \mathcal{O}(-9) \leftarrow 0$$

Construction
$\mathcal{E} = 2\mathcal{O}(-1) \oplus 2\Omega^3(3),\ \mathcal{F} = 2\mathcal{O} \oplus \ker(\Omega^2(2) \oplus \Omega^1(1) \xrightarrow{\psi} \mathcal{O})$
where ψ is a special morphism; see (3.43) for more details
see also (3.43) for a discussion of liaison

Cohomology

$h^i(\mathcal{I}_S(p))$ (rows i, top to bottom; columns p, left to right)

2				
	2	1		
			1	1
				2

Family properties
irreducible, unirational
$\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 50$

11.G1.

Invariants
$d = 11,\ \pi = 12,\ HK = 11,\ K^2 = 2,\ \chi = 4,\ q = 0,\ N_6 = 1$

Classification
$S = S_{min}(p_1)$ non-minimal, general type Horikawa surface
$H = H_{min} - E_1$

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow 12\mathcal{O}(-5) \longleftarrow 19\mathcal{O}(-6) \longleftarrow 10\mathcal{O}(-7) \longleftarrow 2\mathcal{O}(-8) \leftarrow 0$$

Construction
$\mathcal{E} = 3\mathcal{O}(-1) \oplus \Omega^3(3),\ \mathcal{F} = 2\Omega^1(1)$
see (3.49) for more details

Cohomology

$h^i(\mathcal{I}_S(p))$ (rows i, top to bottom; columns p, left to right)

3				
	1			
			2	

Family properties
irreducible, unirational
$\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 52$

11.G2.

Invariants
$d=11,\ \pi=12,\ HK=11,\ K^2=2,\ \chi=4,\ q=0,\ N_6=1$

Classification
$S=S_{min}$ minimal, general type surface
$H=H_{min}$

Syzygies of $\mathcal{I}_S$

$$\begin{array}{cccccccc} & 2\mathcal{O}(-4) & & & & & & \\ & \oplus & & & & & & \\ 0\leftarrow\mathcal{I}_S\leftarrow & 7\mathcal{O}(-5) & \nwarrow & 10\mathcal{O}(-6) & & 3\mathcal{O}(-7) & & \\ & \oplus & & \oplus & \longleftarrow & \oplus & \nwarrow & \\ & \mathcal{O}(-6) & & 3\mathcal{O}(-7) & & 3\mathcal{O}(-8) & & \mathcal{O}(-9)\leftarrow 0 \end{array}$$

Construction
$\mathcal{E}=3\mathcal{O}(-1)\oplus\Omega^3(3),\ \mathcal{F}=\mathcal{O}\oplus\ker(9\mathcal{O}\xrightarrow{\psi}2\mathcal{O}(1))$
where ψ is a general morphism.
see (3.49) for more details

Cohomology

i

3				
	1			
			2	1
				1

p

$h^i(\mathcal{I}_S(p))$

Family properties
irreducible, unirational
$\chi(\mathcal{N}_{S|\mathbb{P}^4})=52$

11.G3.

Invariants
$d=11,\ \pi=12,\ HK=11,\ K^2=2,\ \chi=4,\ q=0,\ N_6=1$

Classification
$S=S_{min}(p_1,p_2)$ non-minimal, general type surface
$H=H_{min}-E_1-E_2$

Syzygies of $\mathcal{I}_S$

$$\begin{array}{cccccc} & 2\mathcal{O}(-4) & & & & \\ & \oplus & & & & \\ 0\leftarrow\mathcal{I}_S\leftarrow & 2\mathcal{O}(-5) & \nwarrow & 3\mathcal{O}(-6) & & \\ & \oplus & & \oplus & \nwarrow & \\ & 4\mathcal{O}(-6) & & 10\mathcal{O}(-7) & & 8\mathcal{O}(-8)\longleftarrow 2\mathcal{O}(-9)\leftarrow 0 \end{array}$$

Construction
$\mathcal{E}=3\mathcal{O}(-1)\oplus\Omega^3(3),\ \mathcal{F}=2\mathcal{O}\oplus\ker(8\mathcal{O}\xrightarrow{\psi}2\mathcal{O}(1))$
where ψ is a general morphism; see (3.49) for more details
see also (3.50) for a discussion of liaison

Cohomology

i

3				
	1			
			2	2
				2

p

$h^i(\mathcal{I}_S(p))$

Family properties
irreducible, unirational
$\chi(\mathcal{N}_{S|\mathbb{P}^4})=52$

11.H.

Invariants
$d=11,\ \pi=12,\ HK=11,\ K^2=8,\ \chi=5,\ q=0,\ N_6=3$

Classification
$S=S_{min}$ minimal, general type surface
$H=H_{min}$

Syzygies of $\mathcal{I}_S$

$$\begin{array}{cccccccc} & \mathcal{O}(-4) & & & & & & \\ & \oplus & & & & & & \\ 0\leftarrow\mathcal{I}_S\leftarrow & 6\mathcal{O}(-5) & \nwarrow & 8\mathcal{O}(-6) & & 2\mathcal{O}(-7) & & \\ & \oplus & & \oplus & \longleftarrow & \oplus & \nwarrow & \\ & 3\mathcal{O}(-6) & & 8\mathcal{O}(-7) & & 7\mathcal{O}(-8) & & 2\mathcal{O}(-9)\leftarrow 0 \end{array}$$

Construction
$\mathcal{E}=4\mathcal{O}(-1),\ \mathcal{F}=\ker(4\mathcal{O}(-1)\oplus 2\mathcal{O}\xrightarrow{\psi}\mathcal{O}(1))$
where ψ is a general morphism.
see (3.52) for more details

Cohomology

i

4				
		1	3	2
				1

p

$h^i(\mathcal{I}_S(p))$

Family properties
irreducible, unirational
$\chi(\mathcal{N}_{S|\mathbb{P}^4})=54$

11.J.

Invariants
$d = 11,\ \pi = 13,\ HK = 13,\ K^2 = 3,\ \chi = 5,\ q = 0,\ N_6 = 1$

Classification
$S = S_{min}(p_1)$ non-minimal, general type surface
$H = H_{min} - E_1$

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{matrix} 3\mathcal{O}(-4) \\ \oplus \\ 2\mathcal{O}(-5) \end{matrix} \longleftarrow \begin{matrix} \mathcal{O}(-5) \\ \oplus \\ 4\mathcal{O}(-6) \end{matrix} \nwarrow \mathcal{O}(-7) \leftarrow 0$$

Construction
$\mathcal{E} = 4\mathcal{O}(-1) \oplus \Omega^3(3),\ \mathcal{F} = 3\mathcal{O} \oplus \Omega^2(2)$
see (3.56) for more details
see also (3.59) for a liaison construction

Cohomology

$h^i(\mathcal{I}_S(p))$ (i vertical, p horizontal)

4				
	1	1		
				3

Family properties
irreducible, unirational
$\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 59$

11.L.

Invariants
$d = 11,\ \pi = 13,\ HK = 13,\ K^2 = 9,\ \chi = 6,\ q = 0,\ N_6 = 0$

Classification
$S = S_{min}$ minimal, general type surface
$H = H_{min}$

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{matrix} 2\mathcal{O}(-4) \\ \oplus \\ 5\mathcal{O}(-5) \end{matrix} \nwarrow 10\mathcal{O}(-6) \longleftarrow 5\mathcal{O}(-7) \longleftarrow \mathcal{O}(-8) \leftarrow 0$$

Construction
$\mathcal{E} = 5\mathcal{O}(-1),\ \mathcal{F} = 2\mathcal{O} \oplus \Omega^1(1)$
see (3.60) for more details
see also (3.60) for the $(4,4)$ liaison with an elliptic quintic scroll

Cohomology

$h^i(\mathcal{I}_S(p))$ (i vertical, p horizontal)

5				
			1	
				2

Family properties
irreducible, unirational
$\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 61$

11.M.

Invariants
$d = 11,\ \pi = 14,\ HK = 15,\ K^2 = 10,\ \chi = 7,\ q = 0,\ N_6 = 0$

Classification
$S = S_{min}$ minimal, general type surface
$H = H_{min}$

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{matrix} \mathcal{O}(-3) \\ \oplus \\ 4\mathcal{O}(-5) \end{matrix} \nwarrow 5\mathcal{O}(-6) \longleftarrow \mathcal{O}(-7) \leftarrow 0$$

Construction
$\mathcal{E} = 6\mathcal{O}(-1),\ \mathcal{F} = \Omega^2(2) \oplus \mathcal{O}(1)$
see (3.68) for more details
see also (3.68) for the $(3,5)$ liaison with a Veronese surface

Cohomology

$h^i(\mathcal{I}_S(p))$ (i vertical, p horizontal)

6				
		1		
			1	5

Family properties
irreducible, unirational
$\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 68$

11.N.

Invariants
$d = 11,\ \pi = 14,\ HK = 15,\ K^2 = 16,\ \chi = 8,\ q = 0,\ N_6 = 0$

Classification
$S = S_{min}$ minimal, general type surface
$H = H_{min}$

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow 4\mathcal{O}(-4) \nwarrow \begin{array}{c} 2\mathcal{O}(-5) \\ \oplus \\ \mathcal{O}(-6) \end{array} \leftarrow 0$$

Construction
$\mathcal{E} = 2\mathcal{O}(-5) \oplus \mathcal{O}(-6),\ \mathcal{F} = 4\mathcal{O}(-4)$
see (3.70) for more details
see also (3.70) for the $(4,4)$ liaison with a Castelnuovo surface

Cohomology

$h^i(\mathcal{I}_S(p))$ (axes: i vertical, p horizontal)

7	1			
				4

Family properties
irreducible, unirational, of dimension 70
which equals $\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 70$

3. Degree 12

12.A.

Invariants
$d = 12,\ \pi = 13,\ HK = 12,\ K^2 = 0,\ \chi = 3,\ q = 0,\ N_6 = 10$

Classification
$S = S_{min}$ minimal, proper elliptic surface
$H = H_{min}$

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{array}{c} 3\mathcal{O}(-5) \\ \oplus \\ 12\mathcal{O}(-6) \end{array} \nwarrow 30\mathcal{O}(-7) \longleftarrow 21\mathcal{O}(-8) \longleftarrow 5\mathcal{O}(-9) \leftarrow 0$$

Construction
$\mathcal{E} = 2\mathcal{O}(-1) \oplus 2\Omega^3(3),\ \mathcal{F} = \ker(15\mathcal{O} \xrightarrow{\psi} 4\mathcal{O}(1))$
where ψ is a special morphism; see (4.1) for details
see also (4.4) for a $(5,5)$ liaison with **13.B.**

Cohomology

$h^i(\mathcal{I}_S(p))$ (axes: i vertical, p horizontal)

2				
	2			
			4	5

Family properties
irreducible, unirational, of dimension 49
which differs from $\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 42$

12.B1.

Invariants
$d = 12,\ \pi = 14,\ HK = 14,\ K^2 = -5,\ \chi = 3,\ q = 0,\ N_6 = 4$

Cohomology

2				
	3	2		
			1	1

$h^i(\mathcal{I}_S(p))$ (axes: i vertical, p horizontal)

Classification
$S = S_{min}(p_1, ..., p_5)$ non-minimal, proper elliptic surface
$H = H_{min} - 2E_1 - \sum_2^5 E_i$

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow 8\mathcal{O}(-5) \nwarrow \begin{array}{c} 7\mathcal{O}(-6) \\ \oplus \\ 4\mathcal{O}(-7) \end{array} \longleftarrow \begin{array}{c} \mathcal{O}(-7) \\ \oplus \\ 4\mathcal{O}(-8) \end{array} \nwarrow \mathcal{O}(-9) \leftarrow 0$$

Construction
$\mathcal{E} = 2\mathcal{O}(-1) \oplus 3\Omega^3(3),\ \mathcal{F} = \ker(2\Omega^2(2) \oplus \Omega^1(1) \xrightarrow{\psi} \mathcal{O})$
where ψ is a general morphism.
see (4.7) for details

Family properties
irreducible, unirational
$\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 47$

12.B2.

Invariants
$d = 12,\ \pi = 14,\ HK = 14,\ K^2 = -5,\ \chi = 3,\ q = 0,\ N_6 = 4$

Cohomology

2				
	3	2		
			1	1

$h^i(\mathcal{I}_S(p))$ (axes: i vertical, p horizontal)

Classification
$S = S_{min}(p_1, ..., p_5)$ non-minimal, proper elliptic surface
$H = H_{min} - \sum_1^5 E_i$

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{array}{c} 8\mathcal{O}(-5) \\ \oplus \\ 2\mathcal{O}(-6) \end{array} \longleftarrow \begin{array}{c} 9\mathcal{O}(-6) \\ \oplus \\ 5\mathcal{O}(-7) \end{array} \longleftarrow \begin{array}{c} 2\mathcal{O}(-7) \\ \oplus \\ 4\mathcal{O}(-8) \end{array} \nwarrow \mathcal{O}(-9) \leftarrow 0$$

Construction
$\mathcal{E} = 2\mathcal{O}(-1) \oplus 3\Omega^3(3),\ \mathcal{F} = \ker(2\Omega^2(2) \oplus \Omega^1(1) \xrightarrow{\psi} \mathcal{O})$
where ψ is a special morphism.
see (4.7) for details

Family properties
irreducible, unirational
$\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 47$

4. Degree 13

13.A1.

Invariants
$d = 13,\ \pi = 16,\ HK = 17,\ K^2 = -17,\ \chi = 1,\ q = 0,\ N_6 = 17$, [DES]

Cohomology

	6	5	1	
				1

i, p

$\mathrm{h}^i(\mathcal{I}_S(p))$

Classification
$S = S_{min}(p_1, \ldots, p_{17})$ non-minimal, Enriques surface
$H = H_{min} - \sum_1^{17} E_i$

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{matrix} 5\mathcal{O}(-5) \\ \oplus \\ \mathcal{O}(-6) \end{matrix} \nwarrow 10\mathcal{O}(-7) \longleftarrow 6\mathcal{O}(-8) \longleftarrow \mathcal{O}(-9) \leftarrow 0$$

Construction
$\mathcal{E} = 16\mathcal{O}$, $\mathcal{F} = \ker(\mathcal{S}yz_2(N) \xrightarrow{\psi} \mathcal{O})$
where ψ is general and N^*
is a generic module with syzygies:
see (5.3) for more details

$$N^* \leftarrow R(4) \nwarrow 9R(2) \nwarrow \begin{matrix} 10R(1) \\ \oplus \\ 16R \end{matrix} \longleftarrow \begin{matrix} R \\ \oplus \\ 36R(-1) \end{matrix} \nwarrow 25R(-2) \longleftarrow 6R(-3) \leftarrow 0$$

Family properties
irreducible, unirational
$\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 38$

13.A2.

Invariants
$d = 13,\ \pi = 16,\ HK = 17,\ K^2 = -17,\ \chi = 1,\ q = 0,\ N_6 = 17$

Cohomology

	6	5	1	
				1

i, p

$\mathrm{h}^i(\mathcal{I}_S(p))$

Classification
$S = S_{min}(p_1, \ldots, p_{17})$ non-minimal, Enriques surface
$H = H_{min} - \sum_1^{17} E_i$

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{matrix} 5\mathcal{O}(-5) \\ \oplus \\ 2\mathcal{O}(-6) \end{matrix} \longleftarrow \begin{matrix} \mathcal{O}(-6) \\ \oplus \\ 10\mathcal{O}(-7) \end{matrix} \nwarrow 6\mathcal{O}(-8) \longleftarrow \mathcal{O}(-9) \leftarrow 0$$

Construction
$\mathcal{E} = 16\mathcal{O}$, $\mathcal{F} = \ker(\mathcal{S}yz_2(N) \xrightarrow{\psi} \mathcal{O})$
where ψ is general and N^*
is a special module with syzygies:
see (5.3) and ff. for more details

$$N^* \leftarrow R(4) \nwarrow 9R(2) \nwarrow \begin{matrix} 10R(1) \\ \oplus \\ 16R \end{matrix} \longleftarrow \begin{matrix} R \\ \oplus \\ 36R(-1) \end{matrix} \nwarrow 25R(-2) \longleftarrow 6R(-3) \leftarrow 0$$

Family properties
irreducible, unirational
$\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 38$

13.B.

Invariants
$d=13,\ \pi=16,\ HK=17,\ K^2=-11,\ \chi=2,\ q=0,\ N_6=10$

Classification
$S=S_{min}(p_1,...,p_{17})$ non-minimal, $K3$ surface
$H=H_{min}-7E_0-\sum_1^{10}E_i$

Syzygies of $\mathcal{I}_S$

$$0\leftarrow\mathcal{I}_S\leftarrow\begin{array}{c}4\mathcal{O}(-5)\\ \oplus\\ 5\mathcal{O}(-6)\end{array}\nwarrow 16\mathcal{O}(-7)\leftarrow 10\mathcal{O}(-8)\leftarrow 2\mathcal{O}(-9)\leftarrow 0$$

Construction
$\mathcal{E}=\mathcal{O}(-1)\oplus 12\mathcal{O},\ \mathcal{F}=\ker(\mathcal{S}yz_2(N)\xrightarrow{\psi}2\mathcal{O})$
where ψ is general and N
has syzygies of type:

$$N\leftarrow 5R(3)\leftarrow 21R(2)\nwarrow\begin{array}{c}30R(1)\\ \oplus\\ 2R\end{array}\longleftarrow\begin{array}{c}12R\\ \oplus\\ 15R(-1)\end{array}\nwarrow 15R(-2)\leftarrow 4R(-3)\leftarrow 0$$

see (4.4) for more details
see also (4.4) for the $(5,5)$ liaison with **12.A.**

Cohomology

$h^i(\mathcal{I}_S(p))$

1				
	5	4		
				2

Family properties
irreducible, unirational
$\chi(\mathcal{N}_{S|\mathbb{P}^4})=40$

5. Degree 14

14.A.

Invariants
$d=14,\ \pi=19,\ HK=22,\ K^2=-15,\ \chi=2,\ q=0,\ N_6=22$

Classification
$S=S_{min}(p_1,...,p_{17})$ non-minimal, $K3$ surface
$H=H_{min}-4E_0-\sum_1^4 2E_i-\sum_5^{14}E_i$

Syzygies of $\mathcal{I}_S$

$$0\leftarrow\mathcal{I}_S\leftarrow\begin{array}{c}4\mathcal{O}(-5)\\ \oplus\\ 4\mathcal{O}(-6)\end{array}\longleftarrow\begin{array}{c}2\mathcal{O}(-6)\\ \oplus\\ 8\mathcal{O}(-7)\end{array}\nwarrow 3\mathcal{O}(-8)\leftarrow 0$$

Construction
$\mathcal{E}=\mathcal{O}(-1)\oplus 15\mathcal{O},\ \mathcal{F}=\mathcal{S}yz_2(N)$
where N^* is a special module with syzygies:

$$N\leftarrow 3R(4)\nwarrow\begin{array}{c}8R(3)\\ \oplus\\ 2R(2)\end{array}\nwarrow\begin{array}{c}4R(2)\\ \oplus\\ 5R(1)\\ \oplus\\ 15R\end{array}\nwarrow 38R(-1)\leftarrow 28R(-2)\leftarrow 7R(-3)\leftarrow 0$$

see (6.1) for more details
see also (6.2) for a liaison construction.

Cohomology

$h^i(\mathcal{I}_S(p))$

1				
	7	7	3	

Family properties
irreducible, unirational, of dimension 39
which differs from $\chi(\mathcal{N}_{S|\mathbb{P}^4})=38$

6. Degree 15

15.A.

Invariants
$d = 15,\ \pi = 21,\ HK = 25,\ K^2 = -25,\ \chi = 0,\ q = 1,\ N_6 = 25$, [ADHPR]

Classification
$S = S_{min}(p_1, ..., p_{25})$ non-minimal, bielliptic surface
$H = H_{min} - \sum_1^{25} E_i$

Cohomology

$h^i(\mathcal{I}_S(p))$

1	10	10	5	

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{matrix} \mathcal{O}(-5) \\ \oplus \\ 10\mathcal{O}(-6) \end{matrix} \nwarrow 15\mathcal{O}(-7) \longleftarrow 5\mathcal{O}(-8) \leftarrow 0$$

Construction
$\mathcal{E} = 20\mathcal{O}$, $\mathcal{F} = \mathcal{S}yz_2(N)$
where N^* is a module with syzygies:
see [ADHPR] for more details

$$N \leftarrow 5R(4) \leftarrow 15R(3) \nwarrow \begin{matrix} 10R(2) \\ \oplus \\ R(1) \\ \oplus \\ 20R \end{matrix} \nwarrow 45R(-1) \leftarrow 30R(-2) \nwarrow \begin{matrix} 5R(-3) \\ \oplus \\ R(-4) \end{matrix} \leftarrow 0$$

Family properties
irreducible, of dimension 25
which equals $\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 25$

15.B1.

Invariants
$d = 15,\ \pi = 21,\ HK = 25,\ K^2 = -25,\ \chi = 0,\ q = 2,\ N_6 = 25$

Classification
$S = S_{min}(p_1, ..., p_{25})$ non-minimal, abelian surface
$H = H_{min} - \sum_1^{25} E_i$

Cohomology

$h^i(\mathcal{I}_S(p))$

1				
2	10	10	5	

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{matrix} \mathcal{O}(-5) \\ \oplus \\ 10\mathcal{O}(-6) \end{matrix} \nwarrow 15\mathcal{O}(-7) \longleftarrow 5\mathcal{O}(-8) \leftarrow 0$$

Construction
$\mathcal{E} = 20\mathcal{O}$, $\mathcal{F} = \mathcal{S}yz_2(N)$
where N^* is a module with syzygies:
see (7.4) for more details

$$N \leftarrow 5R(4) \leftarrow 15R(3) \nwarrow \begin{matrix} 10R(2) \\ \oplus \\ R(1) \\ \oplus \\ 20R \end{matrix} \nwarrow 45R(-1) \leftarrow 30R(-2) \nwarrow \begin{matrix} 5R(-3) \\ \oplus \\ R(-4) \end{matrix} \leftarrow 0$$

Family properties
irreducible, unirational, of dimension 27
which differs from $\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 25$

15.B2.

Invariants
$d = 15,\ \pi = 21,\ HK = 25,\ K^2 = -25,\ \chi = 0,\ q = 2,\ N_6 = 25,$ [HM],[Au2]

Cohomology

1				
2	10	10	5	

$h^i(\mathcal{I}_S(p))$

Classification
$S = S_{min}(p_1, ..., p_{25})$ non-minimal, abelian surface
$H = H_{min} - \sum_1^{25} E_i$

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{array}{c} 3\mathcal{O}(-5) \\ \oplus \\ 5\mathcal{O}(-6) \end{array} \nwarrow 15\mathcal{O}(-7) \leftarrow 10\mathcal{O}(-8) \leftarrow 2\mathcal{O}(-9) \leftarrow 0$$

Construction
$\mathcal{E} = \mathcal{O}(-1) \oplus 15\mathcal{O},\ \mathcal{F} = \ker(\mathcal{S}yz_2(N) \xrightarrow{\psi} 2\mathcal{O}(-1))$
where N^* is a module with syzygies:
see [HM] and [DES] for more details
see also [Au2] for the liaison construction.

$$N \leftarrow 5R(4) \leftarrow 15R(3) \nwarrow \begin{array}{c} 10R(2) \\ \oplus \\ 4R(1) \\ \oplus \\ 15R \end{array} \nwarrow \begin{array}{c} 2R(1) \\ \oplus \\ 35R(-1) \end{array} \leftarrow 20R(-2) \nwarrow 2R(-4) \leftarrow 0$$

Family properties
irreducible, of dimension 27
which differs from $\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 25$

15.C.

Invariants
$d = 15,\ \pi = 22,\ HK = 27,\ K^2 = -6,\ \chi = 4,\ q = 0,\ N_6 = 14$

Cohomology

3				
	7	8	4	

$h^i(\mathcal{I}_S(p))$

Classification
$S = S_{min}(p_1, ..., p_{17})$ non-minimal, general type surface
$H = H_{min} - \sum_0^8 E_i$

Syzygies of $\mathcal{I}_S$

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{array}{c} 2\mathcal{O}(-5) \\ \oplus \\ 7\mathcal{O}(-6) \end{array} \nwarrow 12\mathcal{O}(-7) \longleftarrow 4\mathcal{O}(-8) \leftarrow 0$$

Construction
$\mathcal{E} = 3\mathcal{O}(-1) \oplus 10\mathcal{O},\ \mathcal{F} = \mathcal{S}yz_2(N)$
where N^* is a special module with syzygies:
see $(6, 12)$ for more details
see also $(6, 12)$ for a liaison construction.

$$N \leftarrow 4R(4) \leftarrow 12R(3) \nwarrow \begin{array}{c} 7R(2) \\ \oplus \\ 5R(1) \\ \oplus \\ 10R \end{array} \nwarrow 34R(-1) \leftarrow 27R(-2) \leftarrow 7R(-3) \leftarrow 0$$

Family properties
irreducible, of dimension 38
which equals $\chi(\mathcal{N}_{S|\mathbb{P}^4}) = 38$