Small Schemes and Varieties of Minimal Degree

by

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Abstract: We prove that if $X \subset \mathbb{P}^r$ is any 2-regular scheme (in the sense of Castelnuovo-Mumford) then X is *small*. This means that if L is a linear space and $Y := L \cap X$ is finite, then Y is *linearly independent* in the sense that the dimension of the linear span of Y is $1 + \deg Y$. The converse is true and well-known for finite schemes, but false in general. The main result of this paper is that the converse, "small implies 2-regular", is also true for reduced schemes (algebraic sets). This is proven by means of a delicate geometric analysis, leading to a complete classification: we show that the components of a small algebraic set are varieties of minimal degree, meeting in a particularly simple way. From the classification one can show that if $X \subset \mathbb{P}^r$ is 2-regular, then so is X_{red}, and so also is the projection of X from any point of X.

Our results extend the Del Pezzo-Bertini classification of varieties of minimal degree, the characterization of these as the varieties of regularity 2 by Eisenbud-Goto, and the construction of 2-regular square-free monomial ideals by Fröberg.

Throughout this paper we will work with projective schemes $X \subset \mathbb{P}^r$ over an algebraically closed field k. The (Castelnuovo-Mumford) regularity of $X \subset \mathbb{P}^r$ is a basic homological measure of the complexity of X and its embedding in $\mathbb{P}^r$ that gives a bound for the degrees of the generators of the defining ideal I_X of X and for many other invariants. The only schemes of regularity 1 are the linear spaces; but no classification is known for projective schemes of regularity 2.

In this paper we prove a structure theorem for reduced 2-regular schemes, showing that their irreducible components are varieties of minimal degree and characterizing how these components can meet. We also show that the reduced structure on any 2-regular scheme is 2-regular, and thus we obtain a complete description of the reduced structures on 2-regular schemes. (Since a high Veronese re-embedding of any zero-dimensional scheme is 2-regular, one cannot hope to characterize the isomorphism types of all 2-regular non-reduced schemes.)

Before stating our results we review some basic notions. For any subscheme $X \subset \mathbb{P}^r$ we write $\operatorname{span}(X)$ for the smallest linear subspace of $\mathbb{P}^r$ containing X. Recall that every variety ($\equiv$ reduced irreducible scheme) $X \subset \mathbb{P}^r$ satisfies the condition

$$(*) \qquad \deg(X) \geq 1 + \operatorname{codim}(X, \operatorname{span}(X))$$

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(see for instance Mumford [1976, Corollary 5.13]). We say that the variety $X \subset \mathbb{P}^r$ has *minimal degree* (more precisely, minimal degree in its span) if equality holds. Surfaces of minimal degree were classified by Del Pezzo [1886], and the classification was extended to all dimensions by Bertini [1907] (see Eisenbud-Harris [1987] for a modern account):

Theorem 0.1 *A projective variety of minimal degree in its span is either a linear space, a quadric hypersurface in a linear space, a rational normal scroll, or a cone over the Veronese surface in $\mathbb{P}^5$.*

This classification was extended to equidimensional algebraic sets that are connected in codimension 1 – the ones for which "minimal degree" is a good notion – by Xambó [1981]. For more general algebraic sets it is not clear that there should exist any interesting generalization of the equality in $(*)$ above. Nevertheless, the notion of smallness is just such a generalization.

Varieties of minimal degree were characterized cohomologically by Eisenbud-Goto [1984], and their result offers a way to generalize the hypothesis of the Del Pezzo-Bertini Theorem to all projective schemes. To state their result recall that $X \subset \mathbb{P}^r$ is said to have *regularity* d, or to be d*-regular,* in the sense of Castelnuovo-Mumford, if the ideal sheaf $\mathcal{I}_X$ satisfies $H^i(\mathcal{I}_X(d-i)) = 0$ for all $i > 0$, or equivalently, if the j-th syzygies of the homogeneous ideal I_X are generated in degrees $\leq d + j$ for all $j \geq 0$ (see Eisenbud-Goto [1984], or Eisenbud [2004] for a proof of the equivalence).

Theorem 0.2 *A variety $X \subset \mathbb{P}^r$ has minimal degree in its linear span if and only if X is 2-regular.*

An old argument of Lazarsfeld (see for instance [2004]), recently refined by Sidman [2002], Caviglia [2003], Eisenbud-Green-Hulek-Popescu [2004] and others, shows that if $X \subset \mathbb{P}^r$ is d-regular and Λ is a linear subspace such that $X \cap \Lambda$ has dimension 0, then $X \cap \Lambda$ is also d-regular. When $d = 2$ we can rephrase this geometrically:

We say that a finite scheme $Y \subset \mathbb{P}^r$ is *linearly independent* if the dimension of the linear span of Y is $1 + \deg Y$. We say that a scheme $X \subset \mathbb{P}^r$ is *small* if, for every linear subspace $\Lambda \subset \mathbb{P}^r$ such that $Y = \Lambda \cap X$ is finite, the scheme Y is linearly independent. (An alternative definition of smallness by a more general property of intersections is given in Theorem 2.2.) Lazarsfeld's argument gives:

Proposition 0.3 *Any 2-regular scheme $X \subset \mathbb{P}^r$ is small.*

Our main results are that the converse holds in the reduced case, and that small reduced schemes have a simple inductive classification. To state the classification, we say that a sequence of closed subschemes $X_1, \ldots, X_n \subset \mathbb{P}^r$ is *linearly joined* if, for all $i = 1, \ldots, n-1$, we have

$$(X_1 \cup \ldots \cup X_i) \cap X_{i+1} = \operatorname{span}(X_1 \cup \ldots \cup X_i) \cap \operatorname{span}(X_{i+1}).$$

Theorem 0.4 *Let $X \subset \mathbb{P}^r$ be an algebraic set. The following conditions are equivalent:*
(a) *X is small.*
(b) *X is 2-regular.*
(c) *$X = X_1 \cup \ldots \cup X_n$, where $X_1, \ldots, X_n$ is a linearly joined sequence of varieties of minimal degree.*

The implication (c) $\Rightarrow$ (b) is easy (see Proposition 3.1) while (b) $\Rightarrow$ (a) is a special case of Proposition 0.3 (see also Section 1 for details). Most of this paper is occupied with the proof of the implication (a) $\Rightarrow$ (c), which requires a delicate geometric analysis of the notion of smallness in the style of classical projective geometry. One of the things that makes the argument subtle is the fact that the linearly joined property of a sequence of varieties is strongly dependent on the ordering, as the following example shows.

Example 0.5 Let L_0 be a line in $\mathbb{P}^4$, and let L_1, L_2, L_3 be 3 general lines that meet L_0. The union $X = \bigcup_{i=0}^{3} L_i$ is 2-regular; in fact it is connected in codimension 1, has minimal degree, and is a degeneration of a rational normal quartic curve. As required by Theorem 0.4, X can be written as the union of a linearly joined sequence of varieties L_0, L_1, L_2, L_3, which are trivially of minimal degree in their spans.

On the other hand, the reverse sequence L_3, L_2, L_1, L_0 is not linearly joined. Indeed, the subset $Y = L_3 \cup L_2 \cup L_1$ is not 2-regular: since Y meets the line L_0 in three points, the ideal of Y requires a cubic generator. It is easy to check that there is no enumeration of the components of X as a linearly joined sequence for which the reverse sequence is linearly joined.

In the special case where X is a union of *coordinate* subspaces, the equivalence of parts (b) and (c) of Theorem 0.4 had been proved by Fröberg [1985, 1988] as an application of Stanley-Reisner theory. Unions of coordinate spaces correspond to simplicial complexes. Using earlier results of Dirac [1961] and Fulkerson-Gross [1965], Fröberg showed that a simplicial complex corresponds to a 2-regular set if and only if it is the clique complex of a chordal graph. (We reprove this and give a generalization in Eisenbud-Green-Hulek-Popescu [2004]. See also Herzog, Hibi, and Zheng [2003] for a related path to Dirac's theorem.) The orderings described in part (c) of Theorem 0.4 are called *perfect elimination orderings* in this context. See Blair-Peyton [1993] for a survey.

Properties that are easy to check for algebraic sets satisfying one of the conditions of Theorem 0.4 may be quite obscure for sets satisfying another. Theorem 0.4 has a number of surprising algebraic and geometric consequences based on this observation:

Corollary 0.6 *If $X \subset \mathbb{P}^r$ is a 2-regular algebraic set, then the union of any two irreducible components of X is again 2-regular.*

By Example 0.5 the same cannot be said of a union of three components.

Proof. By Theorem 0.4, the irreducible components of X are of minimal degree in their spans. and thus also 2-regular by Theorem 0.2. From Proposition 3.1 we see that any 2 components are again linearly joined. The result follows by applying Theorem 0.4 once again. ∎

Corollary 0.7 *Let $X \subset \mathbb{P}^r$ be a 2-regular algebraic set. If $p \in X$ is a point and π_p denotes the linear projection from p, then $\pi_p(X) \subset \mathbb{P}^{r-1}$ is 2-regular.*

By Theorem 0.4 a similar statement holds with "small" in place of "2-regular".

Proof. If $X_1, \ldots, X_n$ is a linearly joined sequence of varieties of minimal degree, then $\pi_p(X_1), \ldots, \pi_p(X_n)$ is a linearly joined sequence of varieties of minimal degree. Theorem 0.4 completes the proof. ∎

Corollary 0.8 *If $X \subset \mathbb{P}^r$ is 2-regular, then $X_{\text{red}} \subset \mathbb{P}^r$ is also 2-regular.*

Proof. By Eisenbud-Green-Hulek-Popescu [2004, Theorem 1.6] (see also Theorem 1.2) the scheme $X \subset \mathbb{P}^r$ is small. It follows that X_{red} is also small (Proposition 2.1). By Theorem 0.4, X_{red} is 2-regular. ∎

Corollary 0.9 *Let $X_1, \ldots, X_n \subset \mathbb{P}^r$ be a collection of varieties of minimal degree in their spans. The union $\bigcup_i X_i$ is small if and only if each pair X_i, X_j is linearly joined and the union of the linear spans $\bigcup_i \operatorname{span}(X_i)$ is small.*

Of course a similar statement will hold for 2-regularity in place of smallness. That version is actually one of the key ingredients in the proof of Theorem 0.4.

Proof. Use Theorem 0.4 and Proposition 3.4 ∎

The plan of the paper is as follows. In Sections 1, 2 and 3 we establish basic properties of 2-regular sets, small projective schemes, and linearly joined sequences of projective schemes. Of particular interest is the Bézout type result, Theorem 2.2: If $X \subset \mathbb{P}^r$ is a small subscheme and $\Lambda \subset \mathbb{P}^r$ is any linear space, then the sum of the degrees of the irreducible components (reduced or not, but not embedded) of $X \cap \Lambda$ is bounded by $\operatorname{codim}(X \cap \Lambda, \Lambda) + 1$. The results in these sections are necessary for the proof of Theorem 0.4, which is carried out in Section 4.

The argument of Lazarsfeld showing that plane sections of $X \cap L$ of a 2-regular scheme X are small works even if X is not 2-regular, but only has an ideal generated by quadrics having only linear syzygies for at least $\dim L$ steps. With slightly stronger hypotheses one can prove a little more; for example that that the syzygies of $X \cap L$ come from syzygies of X by restriction. See Theorem 1.2, and Eisenbud-Green-Hulek-Popescu [2004].

It follows from Corollary 0.9 that the condition that an algebraic set $X \subset \mathbb{P}^r$ be small (or 2-regular) has a "local" part, that the X_i of X should be of minimal degree in their spans, and pairwise linearly joined; and a "global", or combinatorial part, that the subspace arrangement $\bigcup_i \operatorname{span}(X_i)$ be small. In Section 5 we will study small subspace arrangements. In particular, we describe the orderings of subspaces that make them into a linearly joined sequence in terms of certain spanning forests of the intersection graph of Y.

Finally, in Section 6 we discuss how to find generators for the ideal of a reduced 2-regular projective scheme. In particular, we prove that any 2-regular union of linear spaces has ideal generated by products of linear forms. The proof of Theorem 6.1 also provides a free resolution for the ideal of a 2-regular algebraic set, and one can ensure that the first two terms of the resolution are minimal, obtaining a formula for the number of generators of the ideal. Some results along this line have also been obtained by Barile and Morales [2000, 2003].

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1 2-regular schemes

The regularity of the ideal sheaf of a closed subscheme $X \subset \mathbb{P}^r$ is ≤ 1 if and only if X is defined by linear forms, so X is a linear subspace in this case.

By contrast, *any* finite scheme can be embedded as a 2-regular scheme. In fact, we see from the definition of regularity that a high Veronese re-embedding of any given embedding of a zero-dimensional scheme is 2-regular. In fact a complete characterization of 2-regular embeddings of zero-dimensional schemes is well-known:

Proposition 1.1 *A zero-dimensional nondegenerate scheme $X \subset \mathbb{P}^r$ is 2-regular if and only if* $\deg X = 1 + \dim \operatorname{span}(X)$.

Proof. The cohomological condition for regularity is equivalent to saying that X imposes $\deg X$ independent conditions on linear forms, that is, the span of X has dimension $\deg X - 1$. ∎

We will often use the fact that a zero-dimensional plane section of a 2-regular scheme is again 2-regular. As described in the introduction, this follows from an argument of Lazarsfeld, and many others have studied it recently. Here is a version sufficient for our purposes in this paper. We say that an ideal $I \subset S$ has *2-linear resolution for at least p steps* if the i-th syzygies of I are generated in degrees $\leq i+1$ for $i = 0, \ldots, p-1$. For example, if I contains no linear forms, this means that the minimal free resolution of I has the form

$$\cdots \oplus S(-i)^{\beta_{p,i}} \longrightarrow S(-p-1)^{\beta_{p-1}} \longrightarrow \cdots \longrightarrow \oplus S(-2)^{\beta_0} \longrightarrow I \longrightarrow 0.$$

We also say in this case that I satisfies property $\mathbf{N}_{2,p}$; this is the terminology used in Eisenbud-Green-Hulek-Popescu [2004].

Theorem 1.2 *Let S be the polynomial ring on $r+1$ variables, and suppose that $X \subset \mathbb{P}^r$ has homogeneous ideal $I_X \subset S$ such that I_X is generated by quadrics and has linear resolution for at least p steps. If $\Lambda \subset \mathbb{P}^r$ is a linear subspace with* $\operatorname{codim}(\Lambda \cap X, \operatorname{span}(\Lambda \cap X)) \leq p-1$, *then $\Lambda \cap X$ is 2-regular. In particular, any zero-dimensional plane section of a 2-regular scheme is 2-regular.* ∎

For an explicit proof see for example Eisenbud-Green-Hulek-Popescu [2004, Th 1.1]. Combining Proposition 1.1 and Theorem 1.2 we get as a corollary the Proposition 0.3 in the introduction:

Corollary 1.3 *Any 2-regular closed scheme $X \subset \mathbb{P}^r$ is small.* ∎

There is a geometric characterization of reduced 2-regular schemes of higher dimension in the Cohen-Macaulay case. By a result of Hartshorne [1962] (see Eisenbud [1995]), connectedness in codimension 1 is a necessary condition for Cohen-Macaulayness. It turns out that for reduced 2-regular algebraic sets they are equivalent.

Theorem 1.4 *Let $X \subset \mathbb{P}^r$ be an equidimensional projective scheme, and let L be the linear span of X. Suppose that X is reduced,and connected in codimension 1.*
(a) $\deg X \geq \operatorname{codim}(X, L) + 1$.
(b) *X is 2-regular if and only if* $\deg X = \operatorname{codim}(X, L) + 1$.
(c) *If the equivalent conditions in part (b) hold, then the homogeneous coordinate ring of X is Cohen-Macaulay.* ∎

Proof. Part (a) of Theorem 1.4 is elementary: the proof works as in the irreducible case (see Hartshorne [1962]) using the connectedness hypothesis to guarantee that the plane section is nondegenerate. The rest is proven in Eisenbud-Goto [1984]. ∎

The following Corollary is Theorem 0.4 in the special case of irreducible or connected-in-codimension 1 algebraic sets:

Corollary 1.5 *Let $X \subset \mathbb{P}^r$ be a projective scheme, and let L be the linear span of X. Suppose that X is reduced, and connected in codimension 1. If X is small then X is 2-regular.* ∎

A well-known regularity criterion that is essentially due to Mumford [1966, Lecture 14] will play a central role. (See also the last section of Eisenbud [2004] for details.)

Theorem 1.6 *A closed subscheme $X \subset \mathbb{P}^r$ is 2-regular if and only if*
(a) *For some (respectively, any) hyperplane $H \subset \mathbb{P}^r$, defined by a linear form that is locally a nonzerodivisor on $\mathcal{O}_X$, the scheme $X \cap H$ is 2-regular, and*
(b) *X is linearly normal; equivalently, the restriction map* $H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \longrightarrow H^0(\mathcal{O}_X(1))$ *is surjective.* ∎

We will say that a scheme $X \subset \mathbb{P}^r$ is the direct sum of schemes $Z_i \subset \mathbb{P}^r$ if, for each i,

$$\operatorname{span}(Z_i) \cap \operatorname{span}(\bigcup_{j \neq i} Z_j) = \emptyset.$$

In other words the underlying vector space of the span of X is the direct sum of the underlying vector spaces of the spans of the Z_i. If this holds, then most questions about $X \subset \mathbb{P}^r$ can be reduced to questions about the $Z_i \subset \operatorname{span}(Z_i)$; for instance the next result will allow us to reduce questions about 2-regularity to the connected case.

Proposition 1.7 *$X \subset \mathbb{P}^r$ is 2-regular if and only if the connected components of X are 2-regular and X is the direct sum of its connected components.*

Proof. The vanishing $H^i(\mathcal{I}_X(2-i)) \cong H^{i-1}(\mathcal{O}_X(2-i)) = 0$, for all $i \geq 2$, is equivalent to $H^{i-1}(\mathcal{O}_Z(2-i)) = 0$, for all $i \geq 2$, and all connected components Z of X. On the other hand we may assume that X is nondegenerate. Then if $X = Z_1 \cup Z_2$ is a disjoint union, the cohomology of the short exact sequence

$$0 \longrightarrow \mathcal{I}_X(1) \longrightarrow \mathcal{I}_{Z_1}(1) \oplus \mathcal{I}_{Z_2}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^r}(1) \longrightarrow 0$$

yields that $H^1(\mathcal{I}_X(1)) = 0$ if and only if both $H^1(\mathcal{I}_{Z_i}(1)) = 0$, for $i = 1, 2$, and $H^0(\mathcal{I}_{Z_1}(1)) \oplus H^0(\mathcal{I}_{Z_2}(1)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^r}(1))$ is an isomorphism. ∎

2 Basic properties of small algebraic sets

The following remarks will be used frequently in the sequel. They follow at once from the definitions.

Proposition 2.1 *Let $X \subset \mathbb{P}^r$ be a small subscheme.*
(*a*) *Any plane section of X is small.*
(*b*) *X_{red} is small.*

We could have defined smallness by a more general property of intersections. Recall that the *geometric degree* of a closed subscheme $Y \subset \mathbb{P}^r$ is defined as the sum of the degrees of the isolated irreducible (not necessarily reduced) components of Y.

Theorem 2.2 *If $X \subset \mathbb{P}^r$ is a small scheme, and $L \subset \mathbb{P}^r$ is a linear space, then the geometric degree of $X \cap L$ is at most $1 + \operatorname{codim}(X \cap L, L)$.*

Does Theorem 2.2 hold with the *arithmetic degree* (sum of all the degrees of isolated *and* embedded components) in place of the geometric degree? See Bayer-Mumford [1993], Sturmfels-Trung-Vogel [1995], or Miyazaki-Vogel-Yanagawa [1997] for the definition and basic properties of such degrees.

Proof. By Proposition 2.1 we can assume $L = \mathbb{P}^r$. We do induction on r, the case $r = 0$ being obvious. Let Y be the union of the zero-dimensional components of X, so that $X = Y \cup Z$ is the disjoint union of Y and a scheme Z whose components have positive dimension. If Y spans $\mathbb{P}^r$, then successively factoring out elements of the socle of $\mathcal{O}_Y$, we see that Y contains a subscheme Y' of length r spanning an $(r-1)$-plane $\Lambda \subset \mathbb{P}^r$. If Z is non-empty then this plane meets Z nontrivially, and thus the sum of the degrees of the components of $X \cap \Lambda$ is $> r = \dim \Lambda + 1 \geq \operatorname{codim}(\Lambda \cap X, \Lambda)$. Since $X \cap \Lambda$ is small, this contradicts the inductive hypothesis and shows that $Z = \emptyset$. Therefore $X = Y$ is finite, and the desired conclusion follows from the definition of smallness.

On the other hand, suppose that Y does not span $\mathbb{P}^r$. If Λ is a general hyperplane containing Y, then Λ meets all the irreducible components Z_i of Z in schemes $Z_i \cap X$ of the same degree as Z_i. Further, the intersection $Z_i \cap Z_j$ of distinct components has dimension strictly less than that of Z_i or Z_j, so the schemes $Z_i \cap Z_j \cap \Lambda$ do not contain any component of $Z \cap \Lambda$. It follows that the sum of the degrees of the components of $Z \cap \Lambda$ is equal to the corresponding sum for Z. By induction on r, the degree of $Y \subset \Lambda$ plus the sum of the degrees of the components of $Z \cap \Lambda$ is bounded by $1 + \operatorname{codim}(X \cap \Lambda, \Lambda) = 1 + \operatorname{codim}(X, \mathbb{P}^r)$. Thus the same bound holds for X. ∎

We isolate a consequence of Theorem 2.2 for use in the proof of the Theorem 0.4:

Corollary 2.3 *Suppose that $Y, Z \subset \mathbb{P}^s$ are disjoint algebraic sets. If $Y \cup Z$ is small then both Y and Z are small.*

Proof. By symmetry it suffices to show that Y is small. Suppose that L is a linear space that meets Y in a finite scheme. Since $L \cap (Y \cup Z)$ is the disjoint union of $L \cap Y$ and $L \cap Z$, the geometric degree of $L \cap (Y \cup Z)$ is at least as great as the length of $L \cap Y$, so we are done by Theorem 2.2. ∎

The conclusion of Theorem 2.2 may be interpreted as a Bézout type theorem in the case of small varieties. A special case of a result of Lazarsfeld says that if $X \subset \mathbb{P}^r$ is a nondegenerate subvariety and $\Lambda \subset \mathbb{P}^r$ is a linear subspace, then the geometric degree of $X \cap \Lambda$ is bounded above by $\deg(X) - \operatorname{codim}(X, \mathbb{P}^r) + \operatorname{codim}(X \cap \Lambda, \Lambda)$ (see Fulton [1984], Example 12.3.5, which also states Lazarsfeld's more general result, or Fulton-Lazarsfeld [1982]). In the case when X is a variety of minimal degree this yields Theorem 2.2.

Next we analyze the irreducible components, and the relative positions of pairs of irreducible components, of small sets. By Proposition 2.1 the same results would apply to the reduced irreducible components of any small projective scheme.

Proposition 2.4 *Let $X \subset \mathbb{P}^r$ be a small algebraic set.*

(*a*) *Any irreducible component X_i of X is a variety of minimal degree in its linear span; that is* $\deg X_i = \dim \operatorname{span}(X_i) - \dim X_i + 1$.

(*b*) *Any two irreducible components X_i and X_j of X are linearly joined; that is, $X_i \cap X_j = \operatorname{span}(X_i) \cap \operatorname{span}(X_j)$.*

Proof. Write $X = \bigcup_i X_i$, where X_i are the irreducible components of X, and let $L_i = \operatorname{span}(X_i)$ denote the linear span of X_i. Any variety X_i in projective space satisfies $\deg X_i \geq \dim \operatorname{span}(X_i) - \dim X_i + 1$, but in our case, X_i is a component of $L_i \cap X$, so the opposite inequality and thus part (*a*) follows by applying Theorem 2.2 to the linear space L_i.

For part (*b*), we first apply Theorem 2.2 to the linear space $L = \operatorname{span}(L_i \cup L_j)$. By construction X_i and X_j are components of $L \cap X$, so the geometric degree of $L \cap X$ is at least $\deg X_i + \deg X_j$, and so Theorem 2.2 yields

$$\deg X_i + \deg X_j \leq \dim L_i + \dim L_j - \dim(L_i \cap L_j) - \max(\dim X_i, \dim X_j) + 1.$$

On the other hand, by part (*a*), $\deg X_i = \dim L_i - \dim X_i + 1$ and we deduce $\dim(L_i \cap L_j) \leq \min(\dim X_i, \dim X_j) - 1$.

Now suppose that $X_i \cap X_j \neq L_i \cap L_j$. If $\dim L_i \cap L_j > 0$, then to simplify we may cut all the schemes concerned by a general hyperplane H. Because X_i is reduced, irreducible and of dimension > 1, the hyperplane section $H \cap X_i$ is again reduced and irreducible (by Bertini's Theorem) and spans $L_i \cap H$. The same holds for X_j. Continuing to take hyperplane sections, we may reduce to the case where the linear space $L_i \cap L_j$ is just a point p.

If now both X_i and X_j contain p, then $X_i \cap X_j = \{p\} = L_i \cap L_j$ as desired. If however X_i does not contain p but X_j does, then $L_i \cap X$ contains a component through p and thus

$$\deg(X \cap L_i) > \deg X_i \geq \dim L_i - \dim X_i + 1 \geq \dim L_i - \dim(X \cap L_i) + 1,$$

contradicting Theorem 2.2. By symmetry, we may therefore assume that neither X_i nor X_j meets $L_i \cap L_j = \{p\}$, and we must derive a contradiction.

Let $\Lambda_k \subset L_k$, for $k = i, j$, be general planes containing p and having $\dim \Lambda_k = \operatorname{codim}(X_k, L_k)$. The scheme $\Lambda_k \cap X_k$ is zero-dimensional. Set $\Lambda = \operatorname{span}(\Lambda_i \cup \Lambda_j)$. We have $\Lambda \cap L_i = \Lambda_i + (\Lambda_j \cap L_i) = \Lambda_i$, and similarly for $\Lambda \cap L_j$, so the components of $\Lambda_k \cap X_k$, $k = i, j$, are also components of $\Lambda \cap X$. Thus the geometric degree of $\Lambda \cap X$ is at least $\deg X_i + \deg X_j$.

On the other hand, since Λ_i and Λ_j meet in a point, the dimension of Λ is $\dim \Lambda_i + \dim \Lambda_j = \operatorname{codim}(X_i, L_i) + \operatorname{codim}(X_j, L_j) = \deg X_i + \deg X_j - 2$, by part (*a*). But then Theorem 2.2 yields that the geometric degree of $\Lambda \cap X$ is at most $\dim \Lambda + 1 = \deg X_i + \deg X_j - 1$, the desired contradiction. This concludes the proof of (*b*). ∎

3 Linearly joined sequences of schemes

A first connection of the notions of smallness and 2-regularity with the notion of linearly joined sequence is provided by the following observations. A linearly joined sequence of schemes $X, Y \subset \mathbb{P}^r$ is linearly joined in either order, so we simply say that the pair is linearly joined.

Proposition 3.1 *Suppose $X_1, X_2 \subset \mathbb{P}^r$ are linearly joined, and set $X = X_1 \cup X_2$.*
(a) X is small $\Rightarrow$ X_1 and X_2 are both small.
(b) X is 2-regular $\Leftrightarrow$ X_1 and X_2 are both 2-regular.

Proof. Part (a) is easy. To prove part (b), write L for the linear space $X_1 \cap X_2$. The result follows from the exact sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{I}_{X_1} \oplus \mathcal{I}_{X_2} \longrightarrow \mathcal{I}_L \longrightarrow 0.$$

(or see the more general statement in part (c) of Theorem 6.1). ∎

Remark 3.2 In the reduced case the converse to part (a) follows from part (b) and Theorem 0.4.

As with 2-regularity, we can reduce questions about linearly joined sequences to the connected case.

Proposition 3.3 *If $X \subset \mathbb{P}^r$ is the union of a linearly joined sequence of irreducible schemes, then X is the direct sum of its connected components.*

Proof. Let $X_1, \ldots, X_n$ be the linearly joined sequence of irreducible components of X. We do induction on n, the case $n = 1$ being trivial.

By induction we may assume that $X' = \bigcup_{i=1}^{n-1} X_i$ is the direct sum of its connected components $Z'_1, \ldots, Z'_s$. Since $X_n \cap X' = \mathrm{span}(X_n) \cap \mathrm{span}(X')$ is a linear space, X_n can meet at most one of the Z'_i. If X_n does not meet any Z'_i, and thus forms a new connected component, then $\mathrm{span}(X_n)$ is disjoint from $\mathrm{span}(X')$, as required. Thus we may assume that X_n meets a unique component, which we may as well call Z'_s, and the connected components of X are

$$Z_1 = Z'_1,\ \ldots\ , Z_{s-1} = Z'_{s-1},\ \text{and}\ Z_s = Z'_s \cup X_n.$$

If X were not the direct sum of the Z_i, then there would be a nontrivial dependence relation of the form $\sum_1^{s-1} p_i + (p_s + q) = 0$ where each p_i is a vector of homogeneous coordinates of a point in $\mathrm{span}(Z'_i)$, or the 0 vector, and q is a vector of homogeneous coordinates of a point in $\mathrm{span}(X_n)$. q is not the 0 vector since $Z'_1, \ldots, Z'_s$ are direct summands, so q must represent a point in $\mathrm{span}(X') \cap \mathrm{span}(X_n) = X' \cap X_n \subset Z'_s$. Since X' is a direct sum of the Z'_i, it follows that $p_1 = \cdots = p_{s-1} = p_s + q = 0$, as required. ∎

Next we show that condition (b) of Proposition 2.4 is exactly the difference between saying that the sequence of schemes $X_1, \ldots, X_n$ is linearly joined and saying that the (same) sequence of their spans is linearly joined.

Proposition 3.4 *Let $X_1, \ldots, X_n \subset \mathbb{P}^r$ be a sequence of closed subschemes and, for each i, let L_i denote the linear span of X_i. The sequence $X_1, \ldots, X_n$ is linearly joined if and only if the sequence $L_1, \ldots, L_n$ is linearly joined and each pair X_i, X_j is linearly joined, for all $i \neq j$.*

Proof. First suppose that the sequence $X_1, \ldots, X_n$ is linearly joined. It follows at once that $L_1, \ldots, L_n$ is linearly joined. By induction on n we may suppose that all pairs X_i, X_j are linearly joined for $i, j < n$, and it suffices to show that $L_j \cap L_n \subset X_j \cap X_n$ for each $j < n$. Since the sequence $X_1, \ldots, X_n$ is linearly joined we have

$$\bigcup_{j<n}(X_j \cap X_n) = (\bigcup_{j<n} X_j) \cap X_n = \operatorname{span}(\bigcup_{j<n} X_j) \cap L_n.$$

Since this is a linear space it must be contained in one of the $X_i \cap X_n$ for some $i < n$. In particular, $L_j \cap L_n \subset X_i \cap X_n$, for all $j < n$, and thus we see that $L_j \cap L_n \subset X_n$. Since, by induction, the pairs X_i, X_j are linearly joined we also have

$$L_j \cap L_n = L_j \cap L_n \cap X_i \cap X_n \subset L_j \cap X_i \subset X_j,$$

completing the argument.

Conversely, suppose that the X_i are pairwise linearly joined and $L_1, \ldots, L_n$ is a linearly joined sequence. By induction, we may assume that $X_1, \ldots, X_{n-1}$ is linearly joined sequence. But

$$\operatorname{span}(\bigcup_{i<n} X_i) \cap L_n = \operatorname{span}(\bigcup_{i<n} L_i) \cap L_n = (\bigcup_{i<n} L_i) \cap L_n = L_j \cap L_n$$

for some $j < n$. Since the pair X_j, X_n is linearly joined we also have $L_j \cap L_n = X_j \cap X_n$, so $\operatorname{span}(\bigcup_{i<n} X_i) \cap L_n = (\bigcup_{i<n} X_i) \cap X_n$ as required. ∎

4 Proof of Theorem 0.4

We already have the tools to dispose of two implications in Theorem 0.4 easily. Theorem 0.2, together with Proposition 3.1 gives the implication $(c) \Rightarrow (b)$. On the other hand, Theorem 1.2 (proved in Eisenbud-Green-Hulek-Popescu [2004]) includes the implication $(b) \Rightarrow (a)$ as a special case.

The last implication, $(a) \Rightarrow (c)$, will occupy us for the rest of this section. Here is an outline: We first prove $(a) \Rightarrow (c)$ in the case where each X_i is a linear space, then we use this case to prove the implication $(b) \Rightarrow (c)$ in general. Finally we will use the implication $(b) \Rightarrow (c)$ to prove $(a) \Rightarrow (c)$.

Proof of $(a) \Rightarrow (c)$ for unions of planes:

To finish the proof of Theorem 0.4 in the case where X is a union of planes, we will project from a general point on a carefully chosen component of X, using the following results.

Proposition 4.1 *Let $V_i \subset V$, $i = 1, \ldots, m$, be distinct linear subspaces, not contained in one another, such that any $m-1$ of them span the ambient space V. Then there exist vectors $u_i \in V_i \setminus \bigcup_{j \neq i} V_j$ such that the collection $u_1, \ldots, u_m$ is linearly dependent.*

Proof. Since the subspaces V_i are distinct and not contained in one another $Z_i = (\cup_{j \neq i} V_j) \cap V_i$ is a union of finitely many proper linear subspaces of V_i, for all $i = 1, \ldots, m$. In particular $Z_i \neq V_i$. Let W denote the kernel of the natural summation map

$$\varphi : V_1 \oplus V_2 \oplus \cdots \oplus V_m \xrightarrow{(v_1, \ldots, v_m) \mapsto \sum_{i=1}^m v_i} V.$$

Observe that if $\pi_i : V_1 \oplus V_2 \oplus \cdots \oplus V_m \to V_i$ denotes the projection on the i-th factor, then $\pi_i(W) = V_i$ by our hypothesis. It follows that $\pi_i^{-1}(Z_i) \neq W$ for all i. Since the ground field is infinite, there exists a vector $w = (v_1, \ldots, v_m) \in W$ such that $\pi_i(w) \notin Z_i$ for all $i = 1, \ldots, m$. In other words, $v_i \in V_i \setminus (\cup_{j \neq i} V_j)$, $i = 1, \ldots, m$ and $v_1 + \cdots + v_m = 0$. ∎

Proposition 4.2 *Let $X = \bigcup_{i=1}^t X_i \subset \mathbb{P}^r$ be a small union of closed subschemes. If $\Lambda_i \subset X_i$ is a linear subspace for each i such that Λ_i does not meet any X_j for $j \neq i$, then the Λ_i are linearly independent and their linear span meets each X_j precisely in Λ_j.*

Proof. Let Λ be the span of the Λ_i. If the Λ_i were dependent, then there would be a set of points $p_i \in \Lambda_i$ that were dependent. Similarly if $\Lambda \cap X_j \neq \Lambda_j$ then there would be a set of points $p_i \in \Lambda_i$, such that the span of the p_i met X_j outside Λ_j. Consequently, to prove either statement, we may assume that $\Lambda_i = \{p_i\}$.

Since $p_i \notin X_j$ for $j \neq i$, there is a component X_i' of $\Lambda \cap X$ containing p_i but none of the p_j. By Theorem 2.2, the sum of the degrees of the X_i' is at most $\operatorname{codim}(X \cap \Lambda, \Lambda) + 1 \leq \dim(\Lambda) + 1$. Since Λ is spanned by the points $p_i \in X_i'$, the dimension of Λ is at most one less than the number of components X_i'. By Theorem 2.2 we conclude that each $X_i' = \{p_i\}$, the p_i are linearly independent, and there are no other points in $X \cap \Lambda$. ∎

Corollary 4.3 *If $X = \bigcup_i X_i \subset \mathbb{P}^r$ is a small union of linear subspaces, then there exists an i such that X_i is not in the linear span of the $\bigcup_{j \neq i} X_j$.*

Proof. If each component X_i were contained in the span of the others, then passing to affine cones we would have a system of subspaces satisfying the hypothesis of Proposition 4.1. The set of points corresponding to the vectors in the conclusion of Proposition 4.1 would violate Proposition 4.2. ∎

Proposition 4.4 *Let $X = \bigcup_{i=1}^n X_i \subset \mathbb{P}^r$ be a small union of linear subspaces. If $p \in X_1$ is a point that is not on the span of $\bigcup_{i=2}^n X_i$, then the image of X under linear projection from p, $\pi_p : \mathbb{P}^r \dashrightarrow \mathbb{P}^{r-1}$, is small.*

Proof. By hypothesis there is a hyperplane $H \subset \mathbb{P}^r$ that contains $\bigcup_{i=2}^n X_i$ but does not contain p. By Proposition 2.1 the union of linear spaces $X' = (H \cap X)_{\mathrm{red}}$ is small, and the projection π_p induces a linear isomorphism $X' \subset H \longrightarrow \pi_p(X) \subset \mathbb{P}^{r-1}$, proving that $\pi_p(X) \subset \mathbb{P}^{r-1}$ is also small. ∎

Proposition 4.5 *Let $X = \bigcup_{i=1}^n X_i \subset \mathbb{P}^r$ be a small union of linear spaces, and let $p \in X_1$ be a point not in the linear span of the union of $X_2, \dots, X_n$. If, for some i, the schemes $\pi_p(X_i)$ and $\pi_p(\bigcup_{j \neq i} X_j)$ are linearly joined, then X_i and $\bigcup_{j \neq i} X_j$ are linearly joined.*

Proof. Let $Z = \bigcup_{j \neq i} X_i$. We must show that if $q \in X_i \cap \mathrm{span}(Z)$ then $q \in Z$. Note that p cannot be in both X_i and $\mathrm{span}(Z)$, so $p \neq q$, and the projection $\pi_p(q)$ is defined.

First suppose $i = 1$. We have $\pi_p(q) \in \pi_p(X_1 \cap \mathrm{span}(Z)) = \pi_p(X_1) \cap \pi_p(\mathrm{span}(Z))$ because $p \in X_1$. Moreover, $\pi_p(\mathrm{span}(Z)) = \mathrm{span}(\pi_p(Z))$, and thus, because $\pi_p(X_1)$ and $\pi_p(Z)$ are linearly joined,

$$\pi_p(q) \in \pi_p(X_1) \cap \mathrm{span}(\pi_p(Z)) = \pi_p(X_1) \cap \pi_p(Z).$$

In particular, $\pi_p(q) \in \pi_p(Z)$. As $q \in \mathrm{span}(Z)$, and π_p is an isomorphism on $\mathrm{span}(Z)$, we get $q \in Z \cap X_1$ as required.

Next suppose $i \neq 1$. Because $p \in \mathrm{span}(Z)$, we may argue as before and obtain

$$\pi_p(q) \in \pi_p(X_i) \cap \pi_p(\mathrm{span}(Z)) = \pi_p(X_i) \cap \pi_p(Z).$$

Thus the line $\mathrm{span}(p, q)$ meets Z in a point q' and meets X_i in a point q''. If $q = q'$, then $q \in Z$, and we are done. Thus we may as well assume that $q \neq q'$, which implies that $p \in \mathrm{span}(q, q')$. By hypothesis $p \notin \mathrm{span}(\bigcup_{i \neq 1} X_i)$, so at least one of the points q, q' must be in X_1. Since $p \in X_1$, this means that both $q, q' \in X_1 \subset Z$; in particular $q \in Z$ as required. ∎

Conclusion of the Proof of $(a) \Rightarrow (c)$ *for a union of linear spaces.* Again, let $X = \bigcup_{i=1}^n X_i \subset \mathbb{P}^r$ be a small union of linear spaces. We do induction on the dimension r of the ambient projective space. After renumbering the components we may assume by Corollary 4.3 that there exists a point $p \in X_1$ that is not in the linear span of $\bigcup_{i \geq 2} X_i$. By Proposition 4.4, the projection $\pi_p(X) \subset \mathbb{P}^{r-1}$ is also small.

By induction on the dimension of the ambient space, there is a permutation of $\{1, \dots, n\}$, such that $\pi_p(X_{\sigma(t+1)})$ is linearly joined to $\bigcup_{i=1}^t \pi_p(X_{\sigma(i)})$ for $t = 1, \dots, n-1$. If $1 \notin \{\sigma(1), \dots, \sigma(t+1)\}$, then π_p is an isomorphism on the linear span of $\bigcup_{i=1}^{t+1} X_{\sigma(i)}$, so X_{t+1} is linearly joined to $\bigcup_{i=1}^t X_{\sigma(i)}$. If on the other hand $1 \in \{\sigma(1), \dots, \sigma(t+1)\}$, then Proposition 4.5 yields the same conclusion. ∎

Proof of $(b) \Rightarrow (c)$ *in the general case.* If X is 2-regular, then by Proposition 2.4 all irreducible components X_i of X are varieties of minimal degree in their spans, and they are pairwise linearly joined. To prove (c) it suffices, by Proposition 3.4, to show that the union of the linear spans of the X_i can be arranged in a linearly joined sequence. The next result shows that this union is 2-regular, and thus reduces the proof to the case we have already treated. The proof will use an induction, and for this we need to know that taking hyperplane sections commutes with taking spans in certain cases.

Lemma 4.6 *Let $Y \subset \mathbb{P}^s$ be a scheme, and H the hyperplane defined by a linear form h. Suppose h is sufficiently general that it is locally a nonzerodivisor on $\mathcal{O}_Y$. If $h^0(\mathcal{O}_Y) = 1$, then* $\mathrm{span}(H \cap Y) = H \cap \mathrm{span}(Y)$.

Proof. The diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^s}) & \xrightarrow{h} & H^0(\mathcal{O}_{\mathbb{P}^s}(1)) & \xrightarrow{\rho} & H^0(\mathcal{O}_H(1)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(\mathcal{O}_Y) & \xrightarrow{h} & H^0(\mathcal{O}_Y(1)) & \longrightarrow & H^0(\mathcal{O}_{Y\cap H}(1)) & \longrightarrow & \cdots
\end{array}$$

has exact rows, and the left-hand vertical map is surjective by the connectedness of Y. By the snake lemma, the restriction ρ induces the isomorphism $H^0(\mathcal{I}_Y(1)) \cong H^0(\mathcal{I}_{Y\cap H,H}(1))$. It follows that $\mathrm{span}(H \cap Y) = H \cap \mathrm{span}(Y)$. ∎

Theorem 4.7 *Let $X = \bigcup_i X_i \subset \mathbb{P}^r$ be a closed subscheme with irreducible components X_i that are Cohen-Macaulay and 2-regular. Let $L_i = \mathrm{span}(X_i)$ and set $Y = \bigcup_i L_i$. If $X_i \cap X_j = L_i \cap L_j$ for each $i \neq j$, then X is 2-regular if and only if Y is 2-regular.*

Proof. We will show that conditions (a) and (b) of Theorem 1.6 hold for X if and only if they hold for Y.

(a): Suppose that X is 2-regular. By Proposition 1.7, X is the direct sum of its connected components, which are also 2-regular. In particular, if X has any zero-dimensional components, they form a direct summand and we may drop them. Thus we may assume that every component of X has dimension ≥ 1.

Let $H \subset \mathbb{P}^r$ be a general hyperplane, so that that $H \cap X$ is 2-regular. Since X_i has dimension ≥ 1 and is Cohen-Macaulay, $h^0(\mathcal{O}_{X_i}) = 1$. By Lemma 4.6, $H \cap L_i$ is the span of $H \cap X_i$.

By Proposition 1.7 again, $H \cap X$ is the direct sum of its connected components, which are also 2-regular, and it suffices to show that the same is true of $H \cap Y$. The connected components of $H \cap Y$ have the form $\bigcup_{i\in I} H \cap L_i$ where I is a minimal set of indices i such that

$$\dim[\bigcup_{i\in I} L_i \cap \bigcup_{j\notin I} L_j] \leq 0.$$

By hypothesis $L_i \cap L_j = X_i \cap X_j$, for all $i \neq j$, so $\bigcup_{i\in I} H \cap X_i$ is a union of connected components of $H \cap X$ (extra components appear when there are 1-dimensional components of X). Since such a union is a direct summand of $H \cap X$, it follows that $H \cap Y$ is a direct sum of its connected components.

Each 1-dimensional irreducible component X_i of X contributes a zero-dimensional scheme spanning a single connected component $H \cap L_i$ of $H \cap Y$ that is a single linear space, and is thus 2-regular. The union of the higher-dimensional irreducible components of X contributes the union of a subset of the connected components of $H \cap X$, which is thus 2-regular. But for an irreducible component X_i of dimension at least 2, $H \cap X_i$ is again irreducible and spans $H \cap L_i$, by Lemma 4.6. Thus we may use induction, and deduce that $H \cap Y$ is 2-regular as required.

The case where $H \cap Y$ is 2-regular is similar. Since it will not be required for the proof of Theorem 0.4, and is a consequence of that result in the case where X is reduced, we omit the details.

(b): Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{O}_Y(1)) \arrow[r] \arrow[d, "\rho_{Y/X}"'] & \bigoplus_i H^0(\mathcal{O}_{L_i}(1)) \arrow[r] \arrow[d, "\oplus\rho_{L_i/X_i}"'] & \bigoplus_{i<j} H^0(\mathcal{O}_{L_i \cap L_j}(1)) \arrow[d, equal, "\oplus\rho_{L_i \cap L_j/X_i \cap X_j}"] \\
0 \arrow[r] & H^0(\mathcal{O}_X(1)) \arrow[r] & \bigoplus_i H^0(\mathcal{O}_{X_i}(1)) \arrow[r] & \bigoplus_{i<j} H^0(\mathcal{O}_{X_i \cap X_j}(1))
\end{tikzcd}$$

where the vertical maps are restrictions. The maps ρ_{L_i/X_i} are isomorphisms because the X_i's are linearly normal and each span L_i. Since $X_i \cap X_j = L_i \cap L_j$ the right hand side vertical map is an equality. Thus the map $\rho_{Y/X}$ is also an isomorphism, so X and Y are either both linearly normal in $\mathbb{P}^r$ or both not. ∎

Proof of Theorem 0.4 continued. We complete the proof of Theorem 0.4 by proving (a) $\Rightarrow$ (b), using induction on $\dim X$. A general hyperplane section of X is reduced by Bertini's Theorem, and by Proposition 2.1, it is small. By induction, it is 2-regular. Thus by Theorem 1.6, it is enough to show that X is linearly normal.

If X were not linearly normal, we could write X as a linear projection of a linearly normal variety Y in some larger projective space $\mathbb{P}^s = \mathbb{P}(H^0(\mathcal{O}_X(1))$ from a linear space $M \subset \mathbb{P}^s$ that is disjoint from Y in such a way that the projection is an isomorphism on Y, but M is contained in the linear span of Y. We first want to show that $Y \cup M$ is small. For this we will need the following basic fact about geometric degree:

Proposition 4.8 *Suppose that $Y \subset \mathbb{P}^s$ is a scheme, and $M \subset \mathbb{P}^s$ is a linear space disjoint from Y. Suppose that the linear projection from M, denoted $\pi_M : \mathbb{P}^s \dashrightarrow \mathbb{P}^r$, induces an isomorphism $Y \to \pi_M(Y)$. If $L \subset \mathbb{P}^s$ is a linear space containing M, then $\pi_M(L) \cap \pi_M(Y) = \pi_M(L \cap Y) \cong L \cap Y$, and the geometric degree of $L \cap Y \subset \mathbb{P}^s$ is the same as the geometric degree of $\pi_M(L) \cap \pi_M(Y)$.*

Proof. First note that because $M \subset L$ the map $\pi_M(Y \cap L) \to \pi_M(Y) \cap \pi_M(L)$ is set-theoretically surjective, so it is enough to check the result locally at a point $q \in Y \cap L$.

Suppose that W is the space of linear forms vanishing on M, and that $V \subset W$ is the space of linear forms vanishing on L. We may identify W with $H^0(\mathcal{O}_{\mathbb{P}^r}(1))$, and under this identification, V is identified with the space of linear forms on $\mathbb{P}^r$ that vanishes on $\pi_M(L)$. With this understanding, the first claim in the statement of the theorem may be written, locally at q, as

$$\mathcal{O}_{Y\cap L,q} \cong \mathcal{O}_{Y,q}/V\mathcal{O}_{Y,q} \cong \mathcal{O}_{\pi_M(Y),\pi_M(q)}/V\mathcal{O}_{\pi_M(Y),\pi_M(q)},$$

which holds because π_M induces an isomorphism $\mathcal{O}_{Y,q} \cong \mathcal{O}_{\pi_M(Y),\pi_M(q)}$.

The statement about geometric degrees follows because all the isomorphisms preserve linear forms. ∎

Proposition 4.9 *Suppose that $Y \subset \mathbb{P}^s$ is an algebraic set, and that $M \subset \mathbb{P}^s$ is a linear space, disjoint from Y, such that the linear projection π_M induces an isomorphism $Y \to \pi_M(Y)$. If $\pi_M(Y)$ is small, then so is $Y \cup M$.*

Proof. Suppose that $\Lambda \subset \mathbb{P}^s$ is a plane that meets $X = Y \cup M$ in a finite set. We must show that $\deg(X \cap \Lambda) \leq 1 + \dim(\Lambda)$. Let $L = \operatorname{span}(\Lambda \cup M)$. As $Y \cap \Lambda$ is a plane section of $Y \cap L$, the degree of $Y \cap \Lambda$ is bounded by the geometric degree of $Y \cap L$. Proposition 4.8 implies that this geometric degree is the same as the geometric degree of $\pi_M(Y \cap L) = \pi_M(Y) \cap \pi_M(L) = \pi_M(Y) \cap \pi_M(\Lambda)$. Since $\pi_M(Y)$ is small by hypothesis, this is bounded by $\dim(\pi_M(\Lambda)) + 1$. Combining this inequality with the obvious $\deg(M \cap \Lambda) + \dim(\pi_M(\Lambda)) = \dim(\Lambda)$, we deduce that $\deg(X \cap \Lambda) \leq 1 + \dim(\Lambda)$ as required. ∎

Proof of Theorem 0.4 continued. From the fact that $Y \cup M$ is small, it follows by Corollary 2.3 that Y is small. As Y is linearly normal by construction, we see from induction on the dimension, Proposition 2.1 (a), and Theorem 1.6 that Y is 2-regular. We want to show next that Y actually coincides with X (i.e. M is empty), and as a first step we show the following result which interesting on its own.

Proposition 4.10 *Let $Y \subset \mathbb{P}^s$ be a 2-regular algebraic set. If $p \in \mathbb{P}^s$ is a point in the span of Y, then there is a plane L containing p such that $L \cap Y$ is finite and $L \cap Y$ spans L.*

Proof. Using the implication (b) $\Rightarrow$ (c) of Theorem 0.4, we see that Y is the union of a linearly joined sequence of irreducible varieties of minimal degree $Y_1, \dots, Y_n$. By Proposition 2.4 (a) and Theorem 1.4 (c), the homogeneous coordinate rings of the irreducible components of Y are Cohen-Macaulay. We will prove, by induction on the dimension of Y, a more general result: if $Y \subset \mathbb{P}^s$ is the union of a sequence of linearly joined irreducible schemes $Y_1, \dots Y_n$ such that each homogeneous coordinate ring S_{Y_i} is Cohen-Macaulay, and no $(Y_i)_{\text{red}}$ is contained in another, then for any point $p \in \operatorname{span}(Y)$, there exists a plane L containing p such that $L \cap Y$ is finite and $L \cap Y$ spans L.

By Proposition 3.3 the problem reduces to the case where Y is connected. If Y is supported at a point the assertion is obvious. Otherwise, each component of Y must have dimension ≥ 1. Under this circumstance, we show that $h^0(\mathcal{O}_Y) = 1$. Since S_{Y_i} has depth at least 2 we have $\bigoplus_m H^0(\mathcal{O}_{Y_i}(m)) \cong S_{Y_i}$, so $h^0(\mathcal{O}_{Y_i}) = 1$ for every i. In general, if $A, B \subset \mathbb{P}^s$ are schemes with $h^0(\mathcal{O}_A) = h^0(\mathcal{O}_B) = 1$, and if $A \cap B \neq \emptyset$, then taking the cohomology of the short exact sequence

$$0 \longrightarrow \mathcal{O}_{A\cup B} \longrightarrow \mathcal{O}_A \oplus \mathcal{O}_B \longrightarrow \mathcal{O}_{A\cap B} \longrightarrow 0$$

gives that $h^0(\mathcal{O}_{A\cup B}) = 1$ also. In particular, $h^0(\mathcal{O}_Y) = 1$ as claimed.

Let H be a general hyperplane containing p, defined by a linear form $x \in H^0(\mathcal{O}_{\mathbb{P}^s}(1))$. Since H cuts properly every component Y_i of Y, $S_{H\cap Y_i} = S_{Y_i}/(x)$ is again Cohen-Macaulay. It follows from the definitions that the sequence $H \cap Y_1, \ldots, H \cap Y_n$ is linearly joined. Since Y is connected and reduces we see that $h^0(\mathcal{O}_Y) = 1$. By Lemma 4.6 H is spanned by $H \cap Y$, and we are done. ∎

Corollary 4.11 *If Y is a 2-regular algebraic set, and Y' is a scheme disjoint from Y that meets the span of Y, then $Y \cup Y'$ is not small.*

Proof. By Proposition 4.10 there is a linear space L that contains a point of Y' and meets Y in a finite scheme that spans $L \cap Y$. Whatever the dimension of $L \cap Y'$ this violates the conclusion of Theorem 2.2, so $Y \cup Y'$ is not small. ∎

Conclusion of the proof of Theorem 0.4. We may apply Corollary 4.11 to the case of the small schemes $Y \cup M$ from Proposition 4.9 and Y to conclude that M is empty! That is, the original small scheme X was linearly normal, and thus 2-regular. This finishes the proof of $(a) \Rightarrow (b)$, and with it the proof of Theorem 0.4. ∎

Remark 4.12 The conclusion of Proposition 4.10 seems to be true for a wide class of schemes Y. We are grateful to Harm Derksen for pointing out that it does not hold for arbitrary schemes, however: Let $M_1 \subset M_2 \subset \mathbb{P}^4$ be a line contained in a 2-plane in $\mathbb{P}^4$, and let p_1, p_2 be general points of M_2. Let $Y = M_1 \cup P_1 \cup P_2$, where P_i is a double point with $(P_i)_{\mathrm{red}} = p_i$ and general tangent vector. It is immediate that Y spans $\mathbb{P}^4$. However, if q is a general point of $\mathbb{P}^4$, then $q \notin \operatorname{span}(M_1 \cup P_1) \cup \operatorname{span}(M_1 \cup P_2) \cup \operatorname{span}(P_1 \cup P_2)$ and so any linear space L containing q and spanned by $L \cap Y$ must contain at least p_1, p_2, and a point of M_1 which is not collinear with p_1, p_2. It follows easily that such an L is all of $\mathbb{P}^4$, and thus $L \cap Y$ is not finite.

On the other hand, it seems that the conclusion of Proposition 4.10 might hold for all reduced schemes. Kristian Ranestad has proven this in characteristic 0. We are grateful to him for allowing us to include it here.

Proposition 4.13 *Let $X \subset \mathbb{P}^s$ be a reduced scheme over an algebraically closed field of characteristic zero. If p is a point in the linear span of X, then there is a linear subspace $L \subset \mathbb{P}^s$ containing p such that $L \cap X$ is finite and contains a set of distinct points that spans L.*

Proof. We may as well assume that X spans $\mathbb{P}^s$. If $p \in X$ or X is finite there is nothing to prove. So let X_0 be a positive dimensional irreducible component of X and let L_0 be the linear span of X_0. Since L_0 has dimension $r > 0$, we may choose distinct points $x_0, \ldots, x_r \in X_0$, and $x_{r+1}, \ldots, x_s \in X \setminus X_0$, such that $x_0, \ldots, x_s$ span $\mathbb{P}^s$.

If $L \subset \mathbb{P}^s$ is a linear space of dimension $< s$ such that L is spanned by some set of distinct points of X and $p \in L$, then we may replace X by $(L \cap X)_{\text{red}}$ and we are done by induction on s. Thus for example we are done if $p \in L_1 = \operatorname{span}\{x_{r+1}, \ldots, x_s\}$. Therefore we may assume that $p \notin L_1$, so the plane $L_2 = \operatorname{span}(p, L_1)$ meets L_0 in a unique point q. If $q \in X$, then since $L_2 = \operatorname{span}\{q, x_{r+1}, \ldots, x_s\}$ and $r \geq 1$, we are again done by induction.

Thus we may suppose $q \notin X$. It follows that the system of hyperplanes containing L_2 has no basepoints on X_0. Since the the base field is of characteristic zero it follows that a general hyperplane H containing L_2 meets X_0 in a reduced set. Since X_0 is reduced and irreducible, Lemma 4.6 shows that $H \cap X_0$ spans $H \cap L_0$. Therefore we may find distinct points $y_1, \ldots, y_r \in H \cap X_0$ such that $y_1, \ldots, y_r, x_{r+1}, \ldots, x_s$ span H. Since $p \in H$ again we are done by induction on s, and the proposition follows. ∎

5 Small subspace arrangements

In this section we will study the combinatorics of the condition for union of linear subspaces to be small. Recall that a subgraph F of a graph G is called a *forest* if F is a disjoint union of *trees* (acyclic connected graphs). A *leaf* of F is a vertex of F connected to at most one other vertex of F. A forest $F \subset G$ *spans* G if F contains all the vertices of G and any two vertices connected by a path in G are connected by a path in F. We say that an ordering of the vertices of G is *compatible with a spanning forest* $F \subset G$ if the smallest vertex in every connected component is a leaf, and the ordering restricts to the natural ordering on the vertices of any path starting from that leaf.

By a *subspace arrangement* we mean a finite union of incomparable linear subspaces in a projective space, say $Y = \bigcup_{i=1}^n L_i \subset \mathbb{P}^r$. We generally do not distinguish between the set of subspaces and their union. To a subspace arrangement Y we associate the weighted graph G_Y, whose vertices are the subspaces L_i of Y, and whose edges join the pairs of subspaces with non-empty intersection. We define the *weight* of the edge (L_i, L_j) to be $1 + \dim(L_i \cap L_j)$, and the weight of a subgraph is the sum of the weights of its edges. We will be interested in the spanning forests of maximal weight in G_Y. To simplify the notation, we give the vertex L_i weight $1 + \dim(L_i)$ and, for any graph G with edge and vertex weights we define the *weighted Euler characteristic* $\chi_w(G)$ to be the sum of the weights of the vertices minus the sum of the weights of the edges. Thus a spanning forest F for G_Y has maximal weight if and only if it has minimal $\chi_w(F)$.

The main result of this section says that smallness and linearly joined sequences of components of Y are features of certain spanning forests in G_Y.

Theorem 5.1 *Let $Y \subset \mathbb{P}^r$ be a subspace arrangement.*

(a) *Y is small if and only if the weighted graph G_Y has a spanning forest F with $\chi_w(F) = 1 + \dim(\mathrm{span}(Y))$, the smallest possible value.*

(b) *If Y is small, then an ordering of the components of Y makes them into a linearly joined sequence if and only if it is compatible with some spanning forest satisfying the equality above.*

We begin with an elementary result that shows the above given value of χ_w is the smallest possible and explains the connection with linearly joined sequences.

Lemma 5.2 *Let $Y = \bigcup_{i=1}^{n} L_i \subset \mathbb{P}^r$ be a subspace arrangement, and let F be a spanning forest of G_Y.*

(a) *Suppose that L_n is a leaf of F. If Y' and F' are obtained from Y and F by removing L_n, then*

$$\chi_w(F) \geq \chi_w(F') + \bigl[\dim(\mathrm{span}(Y)) - \dim(\mathrm{span}(Y'))\bigr],$$

with equality if and only if L_n is linearly joined with Y' and either $L_n \cap Y'$ is empty or $L_n \cap Y' = L_n \cap L_j$, where L_n is connected to L_j in F.

(b) *$\chi_w(F) \geq 1 + \dim(\mathrm{span}(Y))$.*

Proof of Lemma 5.2. Part (a) is elementary, and part (b) follows from part (a) by induction on the number of components. ∎

Proof of Theorem 5.1. We prove parts (a) and (b) together. First suppose that F is a spanning forest with $\chi_w(F) = 1 + \dim(\mathrm{span}(Y))$, and $L_1, \ldots, L_n$ is a compatible ordering of the subspaces in Y.

It follows that L_n is a leaf of F. Let Y' and F' be obtained by deleting L_n from Y and F respectively. By Lemma 5.2 we have

$$\begin{aligned} 1 + \dim(\mathrm{span}(Y')) &\leq \chi_w(F') \\ &\leq \chi_w(F) - \bigl[\dim(\mathrm{span}(Y)) - \dim(\mathrm{span}(Y'))\bigr] \\ &= 1 + \dim(\mathrm{span}(Y')) \end{aligned}$$

so all the equalities hold. Thus L_n is linearly joined to Y' and, by induction on the number of components, $L_1, \ldots, L_n$ is a linearly joined sequence and Y' is small. Using Theorem 0.4 it follows from Proposition 3.1 that Y is small.

Now suppose that Y is small. By Theorem 0.4 (c) we may order the components of Y to form a linearly joined sequence $L_1, \ldots, L_n$. It suffices to find a spanning forest with which this ordering is compatible. We do induction on the number n of components of Y, the case $n = 1$ being trivial.

Since the sequence $L_1, \ldots, L_{n-1}$ is linearly joined, $Y' = \bigcup_{i=1}^{n-1} L_i$ is also small. By induction, $G_{Y'}$ contains a spanning forest F' with $\chi_w(F') = 1 + \dim(\mathrm{span}(Y'))$

such that the ordering $L_1, \ldots, L_{n-1}$ is compatible with F'. Since $Y' \cap L_n = \text{span}(Y') \cap L_n$ is a linear space, and the ground field is infinite, $Y' \cap L_n$ is either empty or it is equal to $L_j \cap L_n$ for some $j < n$. In the first case L_n is a connected component of G_Y, so adjoining it to F' we get a spanning forest F of G_Y In the second case, we may adjoin L_n to F' and connect it to L_j, obtaining a new spanning forest. In either case we get equality in the formula of Lemma 5.2 (a), and the given order is compatible with the forest F. ∎

Theorem 5.1 makes it interesting to understand better which forests have minimal weighted Euler characteristic. If $Y = \bigcup L_i$ is a subspace arrangement, we say that a forest $F \subset G_Y$ satisfies the *clique-intersection property* if whenever $L_1, \ldots, L_j$ form a path in F we have

$$L_1 \cap L_j = L_1 \cap L_2 \cap \cdots \cap L_j.$$

Proposition 5.3 *Let $Y = \bigcup_{i=1}^n L_i \subset \mathbb{P}^r$ be a small subspace arrangement with intersection graph G_Y. A spanning forest $F \subset G_Y$ has $\chi_w(F) = 1 + \dim(\text{span}(Y))$ (or equivalently, maximal weight) if and only if F satisfies the clique-intersection property.*

Proof. Prim's algorithm [1957] (see also Graham-Hell [1985]) shows that a spanning forest F in a weighted connected graph G has minimal $\chi_w(F)$ if and only if, for each edge (x, y) of G, the path in T joining x to y consists of edges (each) of weight $\geq$ the weight of (x, y) (see also Tarjan [1983, pp. 71–72].) In particular, a spanning forest $T \subset G_Y$ satisfying the clique-intersection property must have minimal weighted Euler characteristic. Since Y is small, Lemma 5.2 shows this is $1 + \dim(\text{span}(Y))$.

On the other hand, suppose that F is any spanning forest with $\chi_w(F) = 1 + \dim(\text{span}(Y))$ and $L_1, \ldots, L_j$ are the spaces along a path in F. By Theorem 5.1 the spaces $L_1, \ldots, L_j$ form a part of a linearly joined sequence $L_{-t}, \ldots, L_0, L_1, \ldots, L_j, L_{j+1}, \ldots, L_s$ involving all the components of Y. It follows that all of the $L_i \cap L_j$, for $i < j$ are contained in one $L_h \cap L_j$, with $h < j$. By Prim's algorithm, $\dim(L_h \cap L_j) \leq \dim(L_{j-1} \cap L_j)$ so all the $L_i \cap L_j$ are contained in $L_{j-1} \cap L_j$. Thus by induction on j we get

$$\begin{aligned} L_1 \cap L_j &= L_1 \cap L_{j-1} \cap L_j \\ &= L_1 \cap L_2 \cap \ldots \cap L_{j-1} \cap L_j \end{aligned}$$

as required. ∎

Another way of finding an ordering of the components of a small subspace arrangement Y as a linearly joined sequence is the following: select vertices of G_Y inductively by choosing, at each step, an unselected vertex i_k which has the maximal number of adjacent vertices among the vertices already selected. For a proof see Tarjan-Yannakakis [1984].

The union of all spanning forests of maximal weight of G_Y is the subgraph $H_Y \subset G_Y$ with the same vertices as the intersection graph G_Y, but where L_i and L_j are joined by an edge only when $L_i \cap L_j$ disconnects Y. Indeed, by Proposition 5.3 a maximal weight spanning forest F of G_Y satisfies the clique-intersection property which easily implies that F is actually a spanning forest in H_Y. Conversely, if $L_i \cap L_j \neq \emptyset$ disconnects Y and $L_i = L_{k_0}, L_{k_1}, \ldots, L_{k_r} = L_j$ is the path joining L_i and L_j in a maximal weight spanning forest F of G_Y, then necessarily $L_{k_m} \cap L_{k_{m+1}} = L_i \cap L_j$, for some $m < r$ and the forest F' obtained by replacing in F the edge $(L_{k_m}, L_{k_{m+1}})$ by (L_i, L_j) is also of maximal weight.

6 Equations and syzygies of reduced 2-regular schemes

Next we show how to find generators for the ideal of a reduced 2-regular projective scheme using property (c) of Theorem 0.4. Consider a closed subscheme $X = X' \cup X'' \subset \mathbb{P}^r$, and set $L' = \operatorname{span}(X')$, $L'' = \operatorname{span}(X'')$. In general it is difficult to find generators for the intersection of two ideals, but if X' and X'' are linearly joined (that is, $X' \cap X'' = L' \cap L''$) then we can give minimal generators and a (non-minimal) free resolution of $I_X = I_{X' \cup X''} = I_{X'} \cap I_{X''}$ explicitly from minimal generators and free resolutions for $I_{X'/L'}$ and $I_{X''/L''}$. Our result extends results of Barile and Morales [2000, 2003].

For simplicity we will suppose throughout that $L' \cup L''$ spans the whole ambient space $\mathbb{P}^r$, and leave the reader the easy task to adapt Theorem 6.1 below to the degenerate case. We write $\mu(I)$ for the minimal number of generators of a homogeneous ideal I, and $\operatorname{reg}(I)$ for its regularity.

Theorem 6.1 *Let $X = X' \cup X'' \subset \mathbb{P}^r$ be a nondegenerate closed scheme that is the union of two subschemes X' and X'' with linear spans L' and L'', respectively. Suppose that X' and X'' are linearly joined along $L = L' \cap L''$.*

(a)
$$I_X = \widetilde{I_{X',L'}} + \widetilde{I_{X'',L''}} + I_{L'} \cdot I_{L''},$$
where $\widetilde{I_{X',L'}}$ is any ideal that vanishes on L'' and restricts on L' to the ideal $I_{X',L'}$ of X' in L', and similarly for $\widetilde{I_{X'',L''}}$.

(b) $\mu(I_X) = \mu(I_{X',L'}) + \mu(I_{X'',L''}) + \mu(I_{L'})\mu(I_{L''})$.

(c) $\operatorname{reg}(I_X) = \max\{2, \operatorname{reg}(I_{X'}), \operatorname{reg}(I_{X''})\}$.

The simplest way to construct an ideal $\widetilde{I_{X',L'}}$ as required in part (a) is to choose coordinates in $\mathbb{P}^r$ so that $x_0, \ldots, x_i$ are coordinates for L' while the linear space where $x_0, \ldots, x_i$ vanish is a subspace of L''. Let $I_{X',L'}$ be the ideal of X' in the homogeneous coordinate ring of L'. If we write generators for $I_{X',L'}$ in terms of the coordinates $x_0, \ldots, x_i$, then we may take the same expressions as generators for $\widetilde{I_{X',L'}}$.

Geometrically, this construction amounts to choosing a subspace $L_0'' \subset L''$ complementary to $L = L' \cap L''$, and taking $\widetilde{I_{X',L'}}$ to be the ideal of the cone with base

X' and vertex L''_0. However, this is not the only choice possible: we may perturb the generators given above by any elements of $I_{L'} \cap I_{L''}$. This may even change the codimension of $\widetilde{I_{X',L'}}$. Note that the generators of $I_{X',L'}$ have degree at least 2. Since $(I_{L'} \cap I_{L''})_{\geq 2} = (I_{L'} \cdot I_{L''})_{\geq 2}$ such a perturbation will not, as it must not, change the given form of I_X.

Proof of Theorem 6.1. Since $I_X = I_{X'} \cap I_{X''}$ and $I_{X'} + I_{X''} = I_L$ there is an exact sequence

$$0 \longrightarrow I_X \longrightarrow I_{X'} \oplus I_{X''} \longrightarrow I_{X' \cap X''} \longrightarrow 0.$$

Since we have assumed that $X \subset \operatorname{span}(L' \cup L'') = \mathbb{P}^r$ is nondegenerate, the space of linear forms vanishing on L is the direct sum of the spaces of linear forms vanishing on L' and L''. If we choose minimal free modules F', F'' and surjections $F' \to I_{L'}$ and $F'' \to I_{L''}$ we get a minimal surjection $F' \oplus F'' \to I_L$, and we may write a minimal free resolution of I_L in the form

$$\wedge(F' \oplus F'')_{\geq 1}: \qquad \cdots \longrightarrow \wedge^2(F' \oplus F'') \longrightarrow F \oplus F' \longrightarrow I_L \longrightarrow 0.$$

Let $\widetilde{I_{X',L'}}$ be any ideal that vanishes on L'' and restricts on L' to the ideal of X' in L', and similarly for $\widetilde{I_{X'',L''}}$. Let G'_1 and G''_1 be free modules that map minimally onto $\widetilde{I_{X',L'}}$ and $\widetilde{I_{X'',L''}}$ respectively, and set $H'_1 = F' \oplus G'_1$ and $H''_1 = F'' \oplus G''_1$. Because $\widetilde{I_{X',L'}}$ reduces modulo $I_{L'}$ to $I_{X',L'}$ the induced map $H'_1 \to I_{X'}$ is surjective, and similarly for $H''_1 \to I_{X''}$. Choose now a minimal free resolution

$$\mathbf{H}': \qquad \cdots \longrightarrow H'_2 \longrightarrow H'_1 \longrightarrow I_{X'} \longrightarrow 0.$$

The Koszul complex on the generators of $I_{L'}$ may be written as $(\wedge F')_{\geq 1}$. Considering degrees, we see that each term $\wedge^i F'$ is a summand of the corresponding term H'_i of $\mathbf{H}'$, and we may write $H'_i = \wedge^i F' \oplus G'_i$ for a suitable free module G'_i. Define $\mathbf{H}'', \wedge F''$ and G''_i similarly, and let $a : H'_1 \oplus H''_1 \to I_{X'} \oplus I_{X''}$ be the projection.

We will choose a map of complexes $\phi : \mathbf{H}' \oplus \mathbf{H}'' \to \wedge(F' \oplus F'')_{\geq 1}$ lifting the natural sum map $I_{X'} \oplus I_{X''} \to I_L$:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
(G'_2 \oplus \wedge^2 F') \oplus (G''_2 \oplus \wedge^2 F'') \arrow[r, "\phi_2"] \arrow[d] & \wedge^2 F' \oplus (F' \otimes F'') \oplus \wedge^2 F'' \arrow[d, "\delta"] \\
(G'_1 \oplus F') \oplus (G''_1 \oplus F'') \arrow[r, "\phi_1"] \arrow[d, "a"'] & F' \oplus F'' \arrow[d, "b"] \\
I_{X'} \oplus I_{X''} \arrow[r] & I_L.
\end{tikzcd}$$

We may make this choice so that the restriction of ϕ to the subcomplex $(\wedge F' \oplus \wedge F'')_{\geq 1} \subset \mathbf{H}$ is the canonical injection into $\wedge(F' \oplus F'')_{\geq 1}$.

The mapping cone of ϕ is a complex

$$\cdots \longrightarrow \left[\wedge^2 F' \oplus (F' \otimes F'') \oplus \wedge^2 F''\right] \oplus \left[(G_1' \oplus F') \oplus (G_1'' \oplus F'')\right] \xrightarrow{\delta \oplus \phi_1} F' \oplus F''$$

whose only homology is I_X, occurring at $H_2' \oplus H_2''$.

Since the map ϕ_1 is surjective and ϕ_2 maps $\wedge^2 F' \oplus \wedge^2 F''$ onto the corresponding summands of $\wedge^2(F' \oplus F'') = \wedge^2 F' \oplus (F' \otimes F'') \oplus \wedge^2 F''$, we see that I_X has a free resolution beginning with

$$\cdots (G_2' \oplus \wedge^2 F') \oplus (G_2'' \oplus \wedge^2 F'') \oplus \wedge^3(F' \oplus F'') \longrightarrow F' \otimes F'' \oplus \ker(\phi_1) \longrightarrow I_X \longrightarrow 0.$$

The ideal I_X is embedded in $I_{X'} \oplus I_{X''}$ as the diagonal. Thus the image of $F' \otimes F''$ in I_X is computed by lifting δ along ϕ_1 and then composing with the map $H_1' \oplus H_1'' \to I_{X'} \oplus I_{X''}$, which lands in the kernel of the projection $I_{X'} \oplus I_{X''} \longrightarrow I_{X' \cap X''} = I_L$. Because of our choice of ϕ, the image of this composite is $I_{L'} \cdot I_{L''}$.

Because $\widetilde{I_{X',L'}} \subset I_{L''}$ we may choose $\phi_1|_{G_1'}$ to be a map with image contained in F''. We similarly choose $\phi_1|_{G_1''}$ to map into F'. It follows that $\ker(\phi_1)$ is the direct sum of $\ker(\phi_1|_{G_1' \oplus F''}) = \mathrm{Transpose}(1, -\phi_1|_{G_1'})(G_1')$ and a corresponding copy of G_1''. Since I_X is embedded diagonally in $I_{X'} \oplus I_{X''}$ the image of $\mathrm{Transpose}(1, -\phi_1|_{G_1'})(G_1')$ is $\widetilde{I_{X',L'}}$. The corresponding result for $\widetilde{I_{X'',L''}}$ follows symmetrically, proving the formula (a).

The resolution of I_X that we have just constructed is not in general minimal. However, the syzygies corresponding to elements of $G_2' \oplus G_2''$ must involve the elements of $G_1' \oplus G_1''$. As these elements have degrees ≥ 2, the generators of G_2' and G_2'' must have degrees ≥ 3. Thus the map from $(F' \otimes F'')$ to S sends the generators minimally onto the generators of $I_{L'} \cdot I_{L''}$. Moreover, the generators of G_1' map onto forms that minimally generate the ideal of X' in its span, and similarly for X''. If there were a dependence relation of the form

$$f' + f'' + p = 0$$

where

$$f' \in \mathrm{Im}(G_1'),\ \ f'' \in \mathrm{Im}(G_1'') \text{ and } p \in \mathrm{Im}(F' \otimes F''),$$

then working modulo the equations of L' we see that $f' = 0$ and similarly $f'' = 0$. It follows that $p = 0$ as well, establishing part (b), the desired relation on minimal numbers of generators.

Statement (c) on the regularity follows at once from the form of the above (not necessarily minimal) resolution. ∎

Corollary 6.2 *Let $X = \bigcup_{i=1}^{n} L_i \subset \mathbb{P}^r$ be a nondegenerate 2-regular union of linear subspaces, linearly joined in that order as in Theorem 0.4. If, for each $i \in \{2, \ldots, n\}$, L_i' is a linear complement in L_i for $L_{r_i} \cap L_i$, where* $\operatorname{span}(L_1, \ldots, L_{i-1}) \cap L_i = L_{r_i} \cap L_i$ *with $1 \leq r_i < i$, then*

$$\begin{aligned} I_X &= \sum_{j=2}^{n} I_{\operatorname{span}(L_j, L_{j+1}, \ldots, L_n)} I_{\operatorname{span}(L_1, \ldots, L_{j-1}, L_{j+1}', \ldots, L_n')} \\ &= \sum_{j=2}^{n} I_{\operatorname{span}(L_j, L_{j+1}', \ldots, L_n')} I_{\operatorname{span}(L_1, \ldots, L_{j-1}, L_{j+1}', \ldots, L_n')}. \end{aligned}$$

Proof. The case $n = 2$ is immediate. By induction on n we may assume that the equations of $X' = L_1 \cup \ldots \cup L_{n-1}$ are given by a similar formula with $n-1$ in place of n. Set $X'' = L_n$. In the expression for I_X in Theorem 6.1, the ideal $\widetilde{I_{X''/L''}}$ may be taken to be 0. We may choose $\widetilde{I_{X',L'}}$ to be the ideal of the cone with base X' and vertex L_n', as in the remark after Theorem 6.1. Since taking the ideals of cones commutes with sums and products in an appropriate sense, and the cone over a span of a collection of linear spaces is obtained by taking the span with the vertex, we arrive at the given formula. ∎

Remark 6.3 By the above choice of L_i' we have, for each $i < j$, that

$$\operatorname{span}(L_i, L_{i+1}', \ldots, L_j') = \operatorname{span}(L_i, L_{i+1}, \ldots, L_j).$$

Applying this with $i = 1, j = n$ we see that $L_1, L_2', \ldots, L_n'$ span $\mathbb{P}^r$. Moreover, since

$$\operatorname{span}(L_1, \ldots, L_j) \cap L_{j+1} = L_{r_j} \cap L_{j+1}$$

by hypothesis, we see that L_{j+1}' is disjoint from $\operatorname{span}(L_1, \ldots, L_j)$. Thus if we think of $\mathbb{P}^r$ as lines in a vector space V, then V is the direct sum of spaces corresponding to $L_1, L_2', \ldots, L_n'$. It follows that we may choose variables x_i so that for each j the space $\operatorname{span}(L_{j+1}', \ldots, L_n')$ is defined by the vanishing of an initial segment $x_0, \ldots, x_{i_j}$, and this set of variables are coordinates on $\operatorname{span}(L_1, \ldots, L_j)$. If $Y \subset \operatorname{span}(L_1, \ldots, L_j)$ is a subvariety, then the cone with base Y and vertex $\operatorname{span}(L_{j+1}' \cup \ldots \cup L_n')$ is defined by the equations of Y in $\operatorname{span}(L_1, \ldots, L_j)$ written in these coordinates. Moreover, the cone over this variety with vertex L_{j+2}' is given by the same equations, and so on.

Corollary 6.4 *The homogeneous ideal $I \subset S$ of any 2-regular union of linear spaces is generated by products of pairs of distinct (independent) linear forms.* ∎

Remark 6.5 Motivated by the structure in Corollary 6.4 for the homogeneous ideal of a 2-regular union of linear spaces one may believe that such an ideal can always be obtained from the squarefree monomial ideal of a 2-regular union of coordinate subspaces in a larger space by factoring out a sequence of linear forms that is a regular sequence on every component and any nonempty mutual intersection of irreducible components. Unfortunately this is false as the following example shows: Let $X \subset \mathbb{P}^6$ consist of a $\mathbb{P}^3$ and three general $\mathbb{P}^2$'s sticking out of it, each meeting the $\mathbb{P}^3$ in a line, say denoted as L_i. The set X is small, but the existence a squarefree monomial ideal "inflation" as above would imply that the homogeneous ideal of the union of the three (general) lines L_1, L_2 and L_3 in $\mathbb{P}^3$ would also be generated by products of pairs of linear forms, which is not the case.

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