Problem Set 9 : Graphs

Due 03/30/04

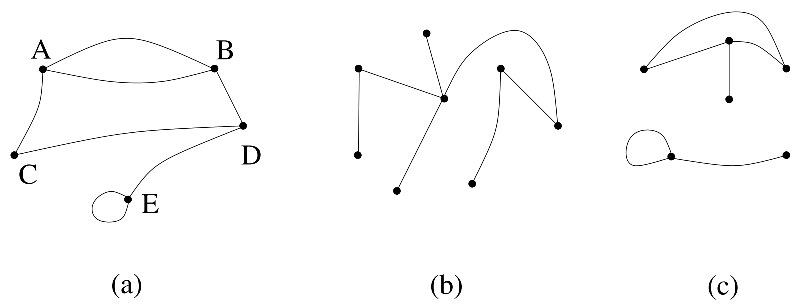
You need to know the following facts for this set of problems.

Graphs. In many problems it is useful to represent objects as points and relationships between objects as lines joining the points.

Example. Suppose that two-way airline routes between the following cities exit: London-Paris, Paris - Rome, Rome - Madrid, London - Rome, Paris - Madrid, Berlin - Oslo, Oslo - Stockholm, Stockholm - Helsinki, Helsinki - Berlin. Is it possible to get from Paris to Stockholm using these routes?
Solution. Draw the cities as labeled points in a plane, and draw the airline connections as the arcs connecting these points. You will see immediately that there is no route connecting Paris to Stockholm.

The mathematical abstraction of such a diagram is called *graph*.

- A *graph* consists of a finite set of points called *vertexes* and a finite number of arcs called *edges* joining some of the vertexes.
  Examples:



- The number of edges attached to each vertex is called the *degree* of that vertex.
  Examples In the above graph (a), vertexes A, B, D, E have degree 3, while vertex C has degree 2.

- A graph is called *connected* if you can connect any two vertexes by a sequence of edges. Thus graphs (a) and (b) are connected, while (c) is not connected.
  Examples Graphs (a) and (b) are connected, whereas graph (c) is not connected.

- A graph is called a *tree* if it contains no "loops" in it. Equivalently, if there is at most one way to connect any two vertexes by edges.
  Example Graph (b) above is a tree, but (a) and (c) are not, since both of them have non-trivial loops in them (in (a), you can find many loops like A→B→D→C→A, or A→B→A or E→E).

1. In any graph, show that the sum of the degrees of all the vertexes is twice the number of edges. (In particular, it follows that this sum must be an even number.)

2. In a town with only 25 telephones, is it possible to connect each telephone by wires to exactly 5 other telephones? (*Hint*: Consider the graph in which vertexes represent the telephones and edges represent wires. Use problem 36.)

3. (a) In a graph, show that the number of vertexes with odd degrees is even.
(b) There are 30 students in a class. Can it happen that 9 of them have 3 friends each (in the class), 11 of them have 4 friends each, and 10 of them have 5 friends each?
(*Hint* : Again, consider the graph in which vertexes represent the students and connect any two friends by an edge.)

4. In a country on the planet Markar, there are 15 towns, each of which is connected by a road to at least 7 others. Prove that you can travel by road from any town to any other town. (possibly through intermediate towns).

5. Can you draw 9 straight line segments in the plane, each of which intersecting exactly 3 others?
(*Hint*: Assume there is such a configuration. Consider the graph in which vertexes represent the line segments and connect two vertexes by an edge if and only if the corresponding line segments intersect.)

6. What is the minimum number of edges in a connected graph with n vertexes ? Prove your claim.