Problem Set 8 : Elementary Number Theory

Due 03/23/04

As usual, think about all problems, solve the first problem and at least one other problem and write them carefully.

You need to know the following facts for this set of problems. In what follows, all the numbers are integers, by which we mean an element of the set $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.

- We say that n *divides* m, or that m is *divisible* by n, if $m = nk$ for some integer k. In this case, we write $n|m$. n is also called a *divisor* of m. For example, $3|12$, $15|135$, $1|n$ and $n|n$ for all integers n.
- An integer $p \geq 2$ is called a *prime* if p and 1 are the only divisors of p. For example, 2, 11, 43 are primes while 35 is not, since both 5 and 7 divide 35. Note that 2 is the only even integer which is also prime.
- **Division Algorithm.** If m and n are arbitrary integers, $n > 0$, there are unique integers q, *quotient*, and r, *remainder*, such that

  $$m = qn + r, \qquad 0 \leq r < n.$$

  q and r are uniquely determined by m and n. It is easy to see that m is divisible by n if and only if the remainder is zero.
- **Congruence.** Given integers n, m and k, we say that n is *congruent to m modulo k* if n and m have the same remainder when we divide them by k. In this case we write $n \equiv m (\text{mod } k)$. For example, $27 \equiv 0 (\text{mod } 3)$, $17 \equiv 5 (\text{mod } 4)$ and $12 \equiv 1002 (\text{mod } 11)$.

  Equivalently, $n \equiv m (\text{mod } k)$ if and only if k divides $n - m$. The relation $\equiv$ has the following property:

  If $n \equiv m (\text{mod } k)$ and $n' \equiv m' (\text{mod } k)$, then

  $$n + n' \equiv m + m' (\text{mod } k)$$

  and also

  $$nn' \equiv mm' (\text{mod } k).$$

1. Let $N = 43 \times 23 - 15 \times 17 + 32 \times 21$, determine:

   (a) the parity of N;

   (b) the units digit of N;

   (c) the remainder when N is divided by 7.

   (Of course, you should do this without actually computing N and explain how you did it.)

2. Prove that if two natural numbers have the same number of copies of each digit in their decimal representations, then they differ by a multiple of 9.

3. What is the rightmost digit of 3^{4798}? What about 7^{33}?
(**Hint:** *Consider congruence modulo 10.*)

4. Prove that for any integer n, $n(n+1)(n+2)$ is divisible by 6.

5. Show that if n is a positive integer then $n^5 - n$ is always divisible by 5.

6. (a) If n is not divisible by 3 can 5n be divisible by 3?
   (b) This time if n is not divisible by 4 can $10n$ be divisible by 4?

7. If p is a prime number $p > 3$ prove that $p^2 - 1$ is divisible by 24.