RESEARCH STATEMENT

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My research focuses on the geometry of 4-manifolds. Just as in dimensions two or three, in dimension four there is a close relation between topological and geometric structures. For $n = 2$ or 3 there is a consensus that a metric of constant sectional curvature should be considered the "best metric" on a given n-manifold. In dimension four it is not so clear what this might mean. I will discuss some canonical metrics that can be defined on 4-manifolds, and a few of their properties. They will appear as minimizing (or just as critical points for) curvature functionals on the space of metrics.

Concerning various canonical metrics, in my thesis I obtain results on:

- the existence and obstructions of Einstein metrics on different differential structures supported by a given topological space;
- complete Einstein metrics on simply-connected noncompact manifolds with infinite homotopy type;
- the moduli space of scalar flat Kähler metrics on $\mathbb{CP}^2\#10\overline{\mathbb{CP}}^2$.

The proofs use the interplay between Riemannian structures and symplectic or complex structures. Another important tool is Seiberg-Witten theory.

While working on an aspect of the Yamabe problem, I came across some interesting results with slightly different flavor.

- Joint with Rasdeaconu, we found a sufficient condition when symplectic surgeries (rational blow downs) can be described as complex operations.

Some of the results of my thesis are explained in two research articles [RS05],[Su05] and other will be explained in future papers.

In More Detail

I begin with a quick introduction to some background material, then I will give a more detailed description of my results and end with present and future projects.

If (M, g) is an oriented Riemannian 4-manifold, the Hodge star operator will yield a decomposition of the space of 2-forms: $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$, where $\Lambda^\pm$ are the $(\pm)$ eigenspaces of $\star$. In consequence the curvature operator $\mathcal{R}$ splits into:

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \overset{\circ}{r} \\ \hline \overset{\circ}{r} & W_- + \frac{s}{12} \end{array} \right)$$

Here s and $\overset{\circ}{r}$ are the scalar curvature and trace-free Ricci, respectively. $W_\pm$ are called the self-dual and anti-self-dual Weyl curvatures.

In what follows I will outline the main results of my thesis, with an emphasis on different classes of distinguished Riemannian metrics.

1. Einstein Metrics

Einstein metrics are one of the best-known classes of distinguished metrics. A metric is called Einstein if its trace-free Ricci curvature $\overset{\circ}{r}$ vanishes.

1.1. **Einstein Metrics on Compact Manifolds.**

The classical obstruction to the existence of Einstein metrics is the *topological* Hitchin-Thorpe Inequality. Only recently, using estimates from Seiberg-Witten equations, LeBrun [LeB01] was able to find obstructions depending on *differential* structures. This leads to the construction of pairs of homeomorphic simply connected manifolds such that one of them admits an Einstein metric, while the other one does not admit any Einstein metric. I have developed methods which prove that a similar statement holds for manifolds with finite cyclic fundamental group. For any fundamental group I have constructed infinitely many classes of manifolds such that each class supports a differential structure that admits an Einstein metric and infinitely many other differential structures that do not admit any Einstein metrics.

The precise statement of the theorem is the following:

Theorem 1.1. [Su05] *For any finite cyclic group $\mathbb{Z}/p\mathbb{Z}$ there exist infinitely many pairs of compact oriented smooth 4-manifolds $(Z_i, M_i^j), i, j \in \mathbb{N}$ satisfying:*

1. *The fundamental group of Z_i and M_i^j is $\mathbb{Z}/p\mathbb{Z}$ for any $i, j \in \mathbb{N}$;*
2. *For i fixed and any j, Z_i and M_i^j are homeomorphic, but no two are diffeomorphic to each other;*
3. *Z_i admits an Einstein metric, while no M_i^j admits any Einstein metrics.*

Moreover their universal covers $\widetilde{Z_i}$ and $\widetilde{M_i^j}$, respectively satisfy:

4. *$\widetilde{M_i^j}$ is diffeomorphic to $\#n\mathbb{CP}^2\#m\overline{\mathbb{CP}}^2$, where $n = b_+(Z_i)$ and $m = b_-(Z_i)$;*
5. *$\widetilde{Z_i}$ and $\widetilde{M_i^j}$ are not diffeomorphic, but become diffeomorphic after connected sum with one copy of $\mathbb{CP}^2$.*

Remark 1.2. The manifolds used in the above theorem are necessarily non-spin. However, using a refined Seiberg-Witten invariant due to Bauer and Furuta, one can also find examples of pairs of spin manifolds (Z, M^j) which satisfy requirements $(1-3)$.

In conditions $(4-5)$ one wants the manifolds $\widetilde{M_i^j}$ to be not only homeomorphic, but diffeomorphic to $\#n\mathbb{CP}^2\#m\overline{\mathbb{CP}}^2$. No symplectic or complex surface can decompose in this way. One way to argue this is that the connected sums $\#n\mathbb{CP}^2\#m\overline{\mathbb{CP}}^2$ $(n > 1)$ have trivial Seiberg-Witten invariants, while for symplectic or simply connected complex surfaces the invariants never vanish. At most we can wish that after taking the connected sum with one copy of $\mathbb{CP}^2$ the new manifold will completely decompose. The manifolds which satisfy the above property are called *almost completely decomposable*, or *acd*. An important step in the proof of Theorem 1.1 was the following:

Theorem 1.3. [Su05]

1. *Bi-double covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along two smooth, transversal curves are almost completely decomposable.*
2. *There exists a constant $n_0 > 0$ such that for all integer pairs (x, y) satisfying the conditions $2x - 6 \leq y \leq 7x$ and $x > n_0$, there is a simply connected, acd, symplectic manifold M such that $y = c_1^2(M)$, and $x = \frac{c_1^2(M)+c_2(M)}{12} = \chi_h(M)$.*

There are no known obstructions to the existence of Einstein metrics on $\#n\mathbb{CP}^2\#m\overline{\mathbb{CP}}^2$, except for the topological Hitchin-Thorpe Inequality. But in some cases, corresponding to the constructions in Theorem 1.1 we can conclude that there are no Einstein metrics invariant under the action of $\mathbb{Z}/p\mathbb{Z}$. To be more precise, one of the examples with the smallest topology would be the following:

Proposition 1.4. *There exists an involution σ on $17\mathbb{CP}^2\#79\overline{\mathbb{CP}^2}$ such that $17\mathbb{CP}^2\#79\overline{\mathbb{CP}^2}$ does not admit any Einstein metric invariant under the involution σ.*

1.2. **Complete Einstein Metrics on Four Manifolds of Infinite Topological Type.**

The problem of the existence of Einstein metrics can also be addressed for non compact manifolds. In [AKL89] Anderson, Kronheimer and LeBrun were able to construct an infinite family of complete Ricci-flat simply connected Kähler manifolds, with infinite second homotopy group. Their metrics are explicit and admit a semi-free isometric S^1-action. Recently there has been a lot of interest manifested in this subject both by mathematicians and physicists. For example Calderbank and Singer studied the same problem for Einstein metrics of negative scalar curvature; their construction uses a free T^2-action on a dense open set of the manifold and some intricate analysis.

However, I think that simply connected manifolds of infinite topological type which admit Einstein metrics of negative scalar curvature are quite common. One can use the existence of Kähler-Einstein metrics on complex surfaces with ample canonical line bundle and infinite fundamental group. Looking at their universal cover I have proved the following:

Theorem 1.5. *There are infinitely many non compact 4-manifolds with infinite second homotopy group which admit complete Einstein metrics of negative scalar curvature.*

2. Algebraic Rational Blow-Down

While I was working on the above problems, I needed to construct new manifolds with non-trivial Seiberg-Witten invariants satisfying various properties. An important class of such examples is the class of symplectic manifolds. The surgeries that I needed to consider were symplectic sum and rational blow down. Symplectic sum and its applications are described by Gompf in [Go95]. Rational blow down was introduced by Fintushel and Stern and consists in removing a neighborhood of a certain chain of spheres of negative self-intersection and gluing in a rational ball. It can be viewed as a generalization of the usual complex blow-down on symplectic manifolds.

In joint work with Răsdeaconu [RS05] we tackled the question: *If the initial manifold is complex, after we rationally blow-down the configuration of spheres, does the new manifold admit a complex structure?* We remark that it is known that there are manifolds which after the rational blow-down do not admit any complex structure. We determined conditions when such surgeries can be realized as complex operations.

The first case of rational blow-down is when we have a symplectic sphere of self-interesection (-4). It consists of removing a neighborhood of the sphere and gluing in the complement of a conic in $\mathbb{CP}^2$. The complex equivalent would be looking at a manifold with simple normal crossing singularity, obtained by identifying the (-4) sphere with the conic in $\mathbb{CP}^2$, and deforming this to a smooth manifold. The complex geometry techniques were introduced by Friedman. This answers our question for some manifolds.

For the general rational blow-down we have to use a different approach. The idea is to collapse the chain of negative curves to a singular point and then smooth the new variety. The techniques were introduced by Manetti [Man01]. We have the following result:

Theorem 2.1. [RS05] *Let G be a finite group acting with only isolated fixed points on a smooth, compact, complex surface S with $H^2(S, \Theta_S) = 0$. If the singularities of S/G are sufficiently nice (i.e. of class T,) then the full rational blowing down $\widetilde{S}$ of the minimal resolution of S/G admits complex structures. Moreover, as a smooth $4-$manifold, $\widetilde{S}$ is oriented diffeomorphic to a 1-parameter $\mathbb{Q}-$Gorenstein smoothing of S/G.*

By full rational blow-down we mean the simultaneous deformation of all the strings of $(-2)-$curves appearing in the minimal resolution of the rational double points and rationally blowing down all of the other chains of exceptional divisors.

An immediate application to symplectic topology is the following. Let W_4 be the simply connected elliptic complex surface, without any multiple fibers and with Euler characteristic $\chi(W_4) = 48$. The elliptic fibration admits nine disjoint sections, which are rational curves of self-intersection (-4). Let $W_{4,n}$ be the manifold obtained by rationally blowing-down n sections, with $1 \leq n \leq 9$. In [Go95] it is showed that for $n = 2, 3, 4, 9$ the manifolds $W_{4,n}$ are diffeomorphic to complex surfaces. By our construction the complex structure is induced naturally and we were also able to include another case $n = 8$.

Theorem 2.2. [RS05] *The manifolds $W_{4,n}, n = 2, 3, 4, 8, 9$ defined above admit a complex structure.*

3. Present and Future Research

Currently, I have plans to develop my research in several directions. Below I have outlined some of the ideas and projects under way.

3.1. **Moduli space of scalar-flat Kähler metrics.** Calabi asked the following question: *in the given Kähler class is there any metric of constant scalar curvature?* In order to study this problem he looked for metrics which minimize the L^2-norm of the scalar curvature. These are called *extremal metrics* of the Kähler class. When the class is unobstructed they are metrics of constant scalar curvature.

Among the Kähler metrics of constant scalar curvature, the flat metrics have a special feature. They are automatically anti-self-dual, i.e. $W_+ = 0$ and they can also be studied via twistor theory. I intend to compare the two spaces of metrics: the moduli space of scalar-flat Kähler metrics, denoted SFK, and the moduli space of anti-self-dual metrics, denoted ASD.

To be more precise, I intend to look at the relations between the two moduli spaces on a special manifold M, namely the smooth four manifold $\mathbb{CP}^2\#10\overline{\mathbb{CP}}^2$. I choose this manifold because 10 is the minimum number of points at which $\mathbb{CP}^2$ needs to be blown-up in order for a scalar-flat Kähler metric to exist.

Rollin and Singer constructed a scalar flat Kähler metric on M that we denote g_{RS}. I have proved the following result:

Theorem 3.1. *A smooth open neighborhood of g_{RS} in $ASD(M)$ consists of scalar flat Kähler metrics.*

There are many other questions that need to be answered. First of all, it is important to discuss whether or not the corresponding connected components coincide. I also intend to investigate the singularities of these spaces. One approach would be through continuity and compactness theorems, in the spirit of Anderson or Tian and Viaclovsky [TV05]. Alternative approach would be via gluing theorems, along submanifold singularities following the lines of Arezzo and Pacard [AP04]. The singularities that I intend to consider are of type $(\mathbb{CP}^1 \times \Delta)/\mathbb{Z}_p$, where Δ is an open ball in $\mathbb{C}$. Scalar-flat Kähler metrics on the resolution of these singularities can be constructed using a toric description of the resolution and following the examples of Joyce [Jo95].

3.2. **Symplectic surgeries as complex operations.** The results obtained with Răsdeaconu [RS05] are part of a more substantial project. We intend to investigate the existence of a complex structure on the remaining manifolds $W_{4,n}, n = 5, 6, 7$. We will also consider a second surgery, called the *symplectic sum*, and determine sufficient conditions when this operation yields complex manifolds. We believe that this will enable us to construct new complex surfaces of general type with small topology.

3.3. **Yamabe invariant.** The Yamabe problem is central in differential geometry. It states that in a conformal class we can always choose a distinguished metric of constant scalar curvature. Using this result, we can define a differential invariant of the manifold by taking the supremum of the unit-volume constant-scalar curvature Riemannian metrics. LeBrun [LeB99] computed this invariant for $4-$manifolds supporting a complex algebraic structure. I will address the same problem for symplectic manifolds. The same arguments show that there is an upper bound on the invariant for manifolds supporting a symplectic structure. However, an important ingredient of LeBrun's program was the existence of Kähler-Einstein metrics, and there is no similar tool in symplectic geometry. We do not have a similar tool on symplectic geometry. But, on particular classes of examples I was able to find an independent proof which computes the invariant. I also computed the invariant for some elementary, symplectic, non-complex manifolds. These results point in the direction that the Yamabe invariant for symplectic manifolds can be computed, and it should be the one expected from the upper bound.

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