SIMILARITY

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Euclidean Geometry can be described as a study of the properties of geometric figures, but not all kinds of conceivable properties. Only the properties which do not change under isometries deserve to be called geometric properties and studied in Euclidian Geometry.

Some geometric properties are invariant under transformations that belongto wider classes. One such class of transformations is similarity transformations. Roughly they can be described as transformations preserving shapes, but changing scales: magnifying or contracting.

The part of Euclidean Geometry that studies the geometric properties unchanged by *similarity transformations* is called the *similarity geometry.* Similarity geometry can be introduced in a number of different ways. The most straightforward of them is based on the notion of ratio of segments.

The similarity geometry is an integral part of Euclidean Geometry. In fact, there is no interesting phenomenon that belong to Euclidean Geometry, but does not survive a rescaling. In this sense, the whole Euclidean Geometry can be considered through the glass of the similarity geometry. Moreover, all the results of Euclidean Geometry concerning relations among distances are obtained using similarity transformations.

However, main notions of the similarity geometry emerge in traditional presentations of Euclidean Geometry (in particular, in the Kiselev textbook) in a very indirect way. Below it is shown how this can be done more naturally, according to the standards of modern mathematics. But first, in Sections 1 - 4, the traditional definitions for ratio of segments and the Euclidean distance are summarized.

1. Ratio of commensurable segments. (See textbook, sections 143-154 for a detailed treatment of this material.)

If a segment CD can be obtained by summing up of n copies of a segment AB, then we say that $\frac{CD}{AB} = n$ and $\frac{AB}{CD} = \frac{1}{n}$.

If for segments AB and CD there exists a segment EF and natural numbers p and q such that $\frac{AB}{EF} = p$ and $\frac{CD}{EF} = q$, then AB and CD are said to be *commensurable,* $\frac{AB}{CD}$ is defined as $\frac{p}{q}$ and the segment EF is called a *common measure* of AB and CD.

The ratio $\frac{AB}{CD}$ *does not depend on the common measure* EF.

This can be deduced from the following two statements.

For any two commensurable segments there exists the greatest common measure.

The greatest common measure can be found by geometric version of the *Euclidean algorithm*. (See textbook, section 146)

If EF is the greatest common measure of segments AB and CD and GH is a common measure of AB and CD, then there exists a natural number n such that $\frac{EF}{GH} = n$.

If a segment AB is longer than a segment CD and these segments are commensurable with a segment EF, then $\frac{AB}{EF} > \frac{CD}{EF}$.

2. Incommensurable segments. There exist segments that are not commensurable. For example, a side and diagonal of a square are not commensurable, see textbook, section 148. Segments that are not commensurable are called *incommensurable*.

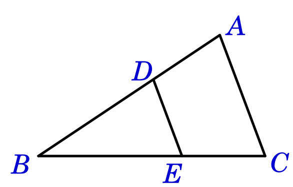
For incommensurable segments AB and CD the ratio $\frac{AB}{CD}$ is defined as the unique real number r such that

- $r < \frac{EF}{CD}$ for any segment EF, which is longer than AB and commensurable with CD;
- $\frac{EF}{CD} < r$ for any segment EF, which is shorter than AB and commensurable with CD.

3. Thales' Theorem. (See Sections 159-160 of the textbook.) *Let ABC be a triangle, D be a point on AB and E be a point on BC. If $DE \parallel AC$, then*

$$\frac{BD}{DA} = \frac{BE}{EC}.$$

□



Corollary. *Under the assumptions of Thales' Theorem,*

$$\frac{BD}{BA} = \frac{BE}{BC} = \frac{DE}{AC}.$$

□

The converse theorem. *Let ABC be a triangle, D be a point on AB and E be a point on BC. If*

$$\frac{BD}{DA} = \frac{BE}{EC},$$

then $DE \parallel AC$.

Proof. Through the point D, draw a line parallel to AC. Let it intersect the side BC at point E'. (We would like to show that $E = E'$.) By the direct theorem, which applies since now we are considering parallel lines,

$$\frac{BD}{DA} = \frac{BE'}{E'C}.$$

But then the hypothesis implies that

$$\frac{BE}{EC} = \frac{BE'}{E'C},$$

and then $E = E'$.

□

4. Distance. If we choose a segment AB and call it the unit, then we can assign to any other segment CD the number $\dfrac{CD}{AB}$, call it the *length* of CD and denote by $|CD|$.

Further, the length $|CD|$ of segment CD is called then the *distance* between points C and D and denote by $\operatorname{dist}(C, D)$. Of course, $\operatorname{dist}(C, D)$ depends on the choice of AB. Define $|CD|$ and $\operatorname{dist}(C, D)$ to be 0 if $C = D$.

The distance between points has the following properties:

- it is symmetric, $\operatorname{dist}(C, D) = \operatorname{dist}(D, C)$ for any points C, D;
- $\operatorname{dist}(C, D) = 0$ if and only if $C = D$;
- triangle inequality, $\operatorname{dist}(C, D) \leq \operatorname{dist}(C, E) + \operatorname{dist}(E, D)$.

5. Definition of similarity transformations. A map S is said to be a *similarity transformation* with *ratio* $k \in \mathbb{R}$, $k \geq 0$, if $|T(A)T(B)| = k|AB|$ for any points A, B in the plane.

Other terms may be used in the same situation: a similarity transformation may be called a *dilation*, or *dilatation*, the ratio may be also called the *coefficient* of the dilation.

General properties of similarity transformations.

1. Any isometry is a similarity transformation with ratio 1.

2. Composition $S \circ T$ of similarity transformations T and S with ratios k and l, respectively, is a similarity transformation with ratio kl.

6. Homothety. An important example of similarity transformation with ratio different from 1 is a homothety.

Definition. Let k be a positive real number, O be a point on the plane. The map which maps O to itself and any point $A \neq O$ to a point B such that the rays OA and OB coincide and $\dfrac{OB}{OA} = k$ is called the *homothety* centered at O with ratio k.

Composition $T \circ S$ of homotheties T and S with the same center and ratios k and l, respectively, is the homothety with the same center and the ratio kl. In particular, any homothety is invertible and the inverse transformation is the homothety with the same center and the inverse ratio.

Theorem 1. *A homothety T with ratio k is a similarity transformation with ratio k.*

Proof. We need to prove that $\frac{T(A)T(B)}{AB} = k$ for any segment AB. Consider, first, the case when O does not belong to the line AB. Then OAB is a triangle, and $OT(A)T(B)$ is also a triangle.

Assume that $k < 1$. Then $T(A)$ belongs to the segment OA, $T(B)$ belongs to OB, and since $\frac{OT(B)}{OB} = \frac{OT(A)}{OA} = k$, the converse to the Thales' theorem implies that $T(A)T(B)$ is parallel to AB. Then, by Corollary of Thales' Theorem, $\frac{T(A)T(B)}{AB} = \frac{OT(A)}{OA} = k$.

If $k > 1$, then A belongs to $OT(A)$, B to $OT(B)$, and the proof is similar.

The case where points A, B, O are collinear is easy, and left as exercise. □

Theorem 2. *Any similarity transformation T with ratio k of the plane is a composition of an isometry and a homothety with ratio k. The center of the homothety can be chosen arbitrarily.*

Proof. Let H be the homothety with ratio k centered at any chosen point O. We would like to show that $T = I \circ H$, where I is an isometry. Notice that the homothety H is invertible: its inverse H^{-1} is the homothety with the same center and coefficient k. Consider the transformation $T \circ H^{-1}$. Then, H^{-1} scales down and T scales up; the composition $T \circ H^{-1}$ of two similarity transformations with coefficients k and k^{-1} is a similarity transformation with coefficient $kk^{-1} = 1$, ie an isometry. Thus, we can set $I = T \circ H^{-1}$.

We have just shown that T can be represented as the composition of an isometry and a homothety, where the homothety is performed first. This argument can be modified to show that T can be also represented as a composition where the isometry is performed before the homothety.

□

Theorem 3. *A similarity transformation of a plane is invertible.*

Proof. By Corollary of Theorem 1, any similarity transformation T is a composition of an isometry and a homothety. A homothety is invertible, as was noticed above. An isometry of the plane is a composition of at most three reflections. Each reflection is invertible, because its composition with itself is the identity. A composition of invertible maps is invertible. □

Corollary. *The transformation inverse to a similarity transformation T with ratio k is a similarity transformation with ratio k^{-1}.*

7. Similar figures. Plane figures F_1 and F_2 are said to be *similar* if there exists a similarity transformation T such that $T(F_1) = F_2$.

Any two congruent figures are similar. In particular, any two lines are congruent and hence similar, any two rays are congruent and hence similar.

Segments are not necessarily congruent, but nonetheless any two segments are similar.

Theorem 4. *1) A figure similar to a segment is a segment, i.e. any similarity transformation maps segments to segments.*

2) Any segment can be mapped to any other by a similarity transformation, i.e. any two segments are similar.

Proof. 1) Let AB be the given segment, S the similarity transformation. By previous theorem, we can write $S = I \circ H_A$, where H_A is a homothety with center A, and I is an isometry. Because the segment AB emanates from the center of homothety, it is clear that H_A maps AB to a segment. (Note that this is far from obvious if the center of homothety lies away from the segment!) Since we know that isometries map segments to segments, we'll still get a segment after applying I.

2) Given segments AB and $A'B'$, we can find an isometry mapping A' to A, and B' to a point on the ray AB. (First find a translation mapping A' to A, and then rotate around the point $A = A'$ match rays AB and $A'B'$.) Then find a homothety with center $A = A'$ mapping one of the segments to the other one.

□

Theorem 5. *1) A figure similar to a circle is a circle, i.e. any similarity transformation maps circles to circles.*

2) Any circle can be mapped to any other by a similarity transformation, i.e. any two circles are similar.

Proof. Exercise. □

Theorem 6. *1) A figure similar to an angle is an angle.*

2)Two angles are similar if and only they are congruent.

Proof. Exercise. □

8. Similarity tests for triangles.

Theorem 7 (AA-test)***.*** *If in triangles ABC and $A'B'C'$ the angles $\angle A$, $\angle A'$ are congruent and angles $\angle B$, $\angle B'$ are congruent, then $\triangle ABC$ is similar to $\triangle A'B'C'$.*

Proof. Without loss of generality we may assume that $A'B'$ is shorter than AB. Find a point D on AB such that $|BD| = |B'A'|$. Draw a segment DE parallel to AC. By ASA test for congruence of triangles, $\triangle A'B'C'$ is congruent to $\triangle DBE$. By Corollary of Thales' Theorem, $\frac{DB}{AB} = \frac{BE}{BC}$. Hence, the homothety centered at B with ratio $\frac{DB}{AB}$ maps $\triangle ABC$ onto $\triangle DBE$. □

Theorem 8 (SAS-test)**.** *If in triangles ABC and $A'B'C'$ the angles $\angle A$, $\angle A'$ are congruent and*
$$\frac{A'B'}{AB} = \frac{A'C'}{AC}$$
then $\triangle ABC$ is similar to $\triangle A'B'C'$.

Proof. Exercise. ☐

Theorem 9 (SSS-test)**.** *If in triangles ABC and $A'B'C'$*
$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA}$$
then $\triangle ABC$ is similar to $\triangle A'B'C'$.

Proof. Without loss of generality we may assume that $A'B'$ is shorter than AB. Find a point D on AB such that $|BD| = |B'A'|$. Draw a segment DE parallel to AC. By Corollary of Thales' Theorem, $\frac{DB}{AB} = \frac{BE}{BC} = \frac{DE}{AC}$. Therefore $|BE| = |B'C'|$ and $|DE| = |A'C'|$. By SSS test for congruence of triangles, $\triangle A'B'C'$ is congruent to $\triangle DBE$. The homothety centered at B with ratio $\frac{DB}{AB}$ maps $\triangle ABC$ onto $\triangle DBE$. ☐

***Theorem 10*.** *Conversely, suppose that triangles ABC and $A'B'C$ are similar, i.e. there exists a similarity transformation S mapping one triangle to the other. For concreteness, we assume that $S(A) = A'$, $S(B) = B'$, $S(C) = C'$. Then $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, and*
$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA}.$$

Proof. We can represent S as the composition of a homothety H_A centered at A and an isometry, $S = I \circ H_A$. Since the center of H_A is one of the vertices of the triangle ABC, this triangle and its image, the triangle $H_A(A)H_A(B)H_A(C)$ are positioned as in Thales' theorem, so the corresponding angles of $\triangle ABC$ and $\triangle H_A(A)H_A(B)H_A(C)$ are equal, and the sides are proportional. On the other hand, the isometry I maps $\triangle H_A(A)H_A(B)H_A(C)$ to $\triangle A'B'C'$, so the latter two triangles are congruent. ☐