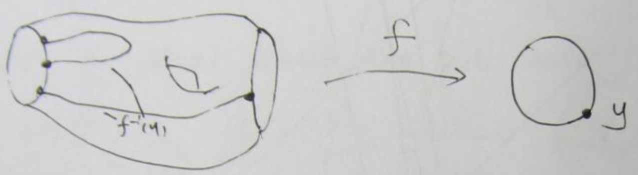
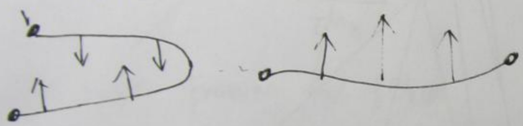
12/09 By Keren Wang

Given a smooth $f: M \to S^p$, $y \in S^p$ regular.

Gives a framing for $f^{-1}(y) \subset M$



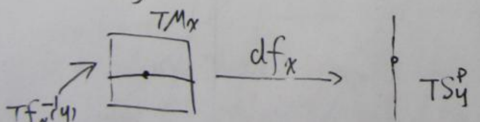
we want



① choose positive basis $\mathcal{B}$ for TS^p_y

② df_x: For $x \in M$. $f(x) = y$, want basis for TM_x. That "moves nicely" along $f^{-1}(y)$.

Look at $df: TM_x \to TS^p$

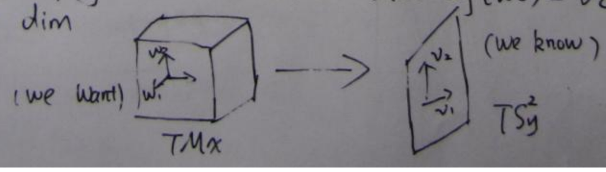


df send $Tf^{-1}(y)_x$ To $0 \subset TS^p_y$. So df is an isomorphism of $Tf^{-1}(y)_x^{\perp}$ to TS^p. ($Tf^{-1}(y)^{\perp}$ means the part of TM which is $\perp$ to $Tf^{-1}(y)$)

$\mathcal{B} = (v_1, \dots v_p)$ for TS^p

For each $v_i \in TS^p$, get $w_i \in TM_x$, so that $df(w_i) = v_i$

ex. In higher dim

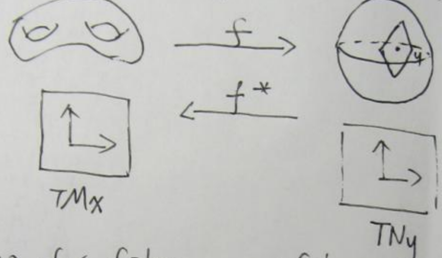


At x, pulling back B for TS^p at y to get basis for $Tf^{-1}(y)_x^{\perp}$.
let B' be the basis.
$B' = f_p^* B$

Aside: $f: M \to N$, smooth, can pull back TN_y to get sth in TM_x

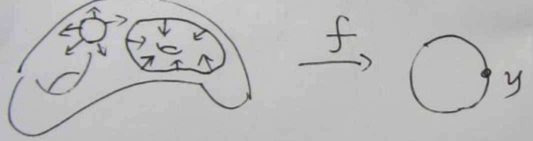
$M \xrightarrow{f} N$

$TM \xleftarrow{f^*} TN$



This gives framing for $f^{-1}(y)$, $(f^{-1}(y), f^*B)$

Def: A framed manifold for $f: M \to S^p$, made this way $(f^{-1}(y), f^*(B))$ is a pontryagin manifold for f.



Thm A: If y, y' are regular values for $f: M \to S^p$ and B, B' are basis for S^p, then $(f^{-1}(y), f^*B)$ is frame cobordant with $(f^{-1}(y'), f^*B')$

Note: (~~For~~ Frame coborant is normally stronger than homotopy. For sphere it's same)

Thm B. Let $f: M \to S^p$ and $g: M \to S^p$ smooth, then $f \overset{\text{smooth}}{\simeq} g$
$\iff (f^{-1}(y), f^*B)$ is frame coborant to $(g^{-1}(y), g^*B')$

Thm C. Any compact framed submanifold (N, B).. (N codim p in M), curves as a pointryagin manifold of M, $f: M \to S^p$.

Lemma: ~~Ca~~ We can choose a smooth path from B to B', in TS^p_y.
With $GL^+(p, \mathbb{R})$. (invertible $p \times p$ matrices, $\det > 0$).