12/5 (?)
6pm.

__We know__

- if we have
  $f: M \to N$
  $g: M \to N$ (M, N are same dimension)

  and $f \sim g$
  $\Rightarrow \deg(f) = \deg(g)$

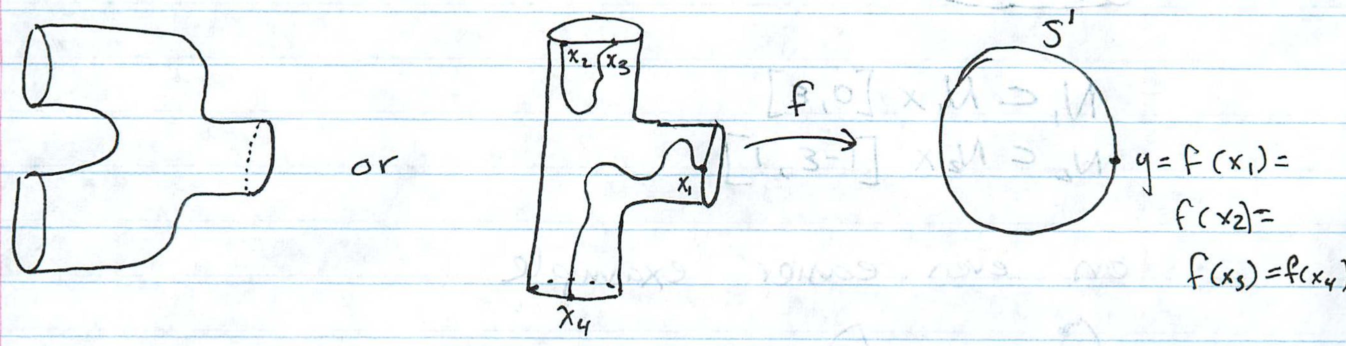
__Goal__: __Thrm__ (hopf)

M oriented, compact, $\partial M = \phi$, $\dim M = n$

$f, g : M \to S^k$

$f \sim g \Leftrightarrow \deg(f) = \deg(g)$

Motivating example

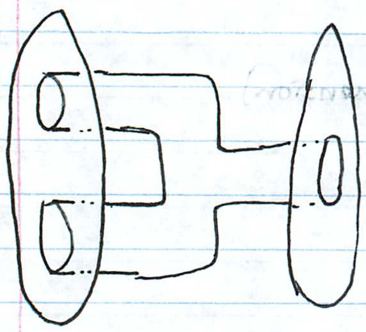


Stuff inside "cancells out" if deg f on ends is the same.

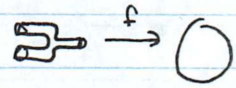
n-m lemma (inverse image of y is a connected 1 manifold with boundary)

inside is "controlled" by ends
(or can understand inside by understanding the outsides
all of the rest is same up to an isomorphism)

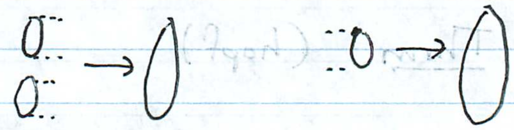
So whats going on here?



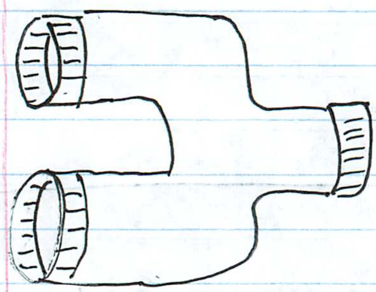
$f: X \to S^1$ w/ $\partial X = N_1 \cup N_2$



$N_1 \to S^1$ $\quad$ $N_2 \to S^1$



can "fatten" N_1 and N_2 into a collar at end of X

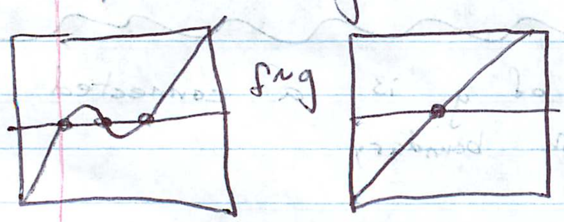


$N_1 \subset N_1 \times [0, \varepsilon]$

$N_2 \subset N_2 \times [1-\varepsilon, 1]$

an even easier example

$f: \overset{S^1}{\overset{=}{N}} \to S^1$ $\quad$ $g: \overset{S^1}{\overset{=}{N}} \to S^1$



Now I can "fill in" with a cylander

to get our homotopy
$F: S^1 \times [0 \times 1] \to S^1$
w/ $F(x,0) = f(x)$
and $F(x,1) = g(x)$

here we can take $\varepsilon = 1$ but maybe stuff inside is more complicated

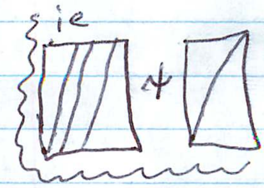
Another way to think about this:

Homotopy is the Entire story

Cobortism is once upon a time ...
(Stuff happened)
... they lived happily ever after
we just need to make sure "stuff that happened" isnt too bad and the beginning and end have some allignment (Just like in homotopy $\deg(f) = \deg(g)$).

2 (sub manifolds) $N_1, N_2 \subset M$ are cobordant if we can "fatten them"
ie) $N_1 \times [0,\varepsilon] \subset M \times [0,1]$
$N_2 \times [1-\varepsilon, 1] \subset M \times [0,1]$
and these can be Embeded into
$X \subset M \times [0,1]$ (compact)
w/ $\partial X = N_1 \times \{0\} \cup N_2 \times \{1\}$
$X \cap (M \times \{0\} \cup M \times \{1\}) = \partial X$

ie



this is saying $N_1 \cup N_2$ are boundary of X and can homotope $N_1 \to N_2$ (via $M \times [0,1]$)

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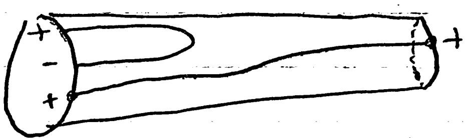
here X is a cobordism between N_1 and N_2

Cobordent means togeather (Co) they make a boundary (bardont)

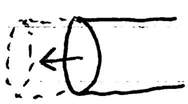
(back to homotopy)



Need orientations on N_1 and N_2 in order to make sign on degree

Such a choice of orientation is a framing

given $N \subset M$



a framing is smooth assignment of basis for,

$$(TN_x)^{\perp} \subset TM_x$$

(ie smooth choice of Normal)

as x moves in N

if basis is B, (N, B) is the framed (Sub)-manifold