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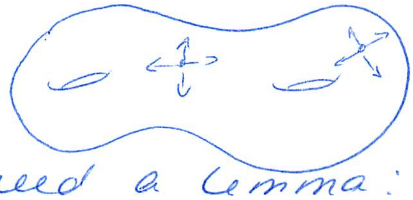
11/30/11

MAT 364
Notes

Poincare-hopf thm:
a vector field on some manifold m, compact, no boundary
then $\chi(m) = \sum_{\substack{z \in \text{zeros} \\ \text{of v.f}}} i(z)$

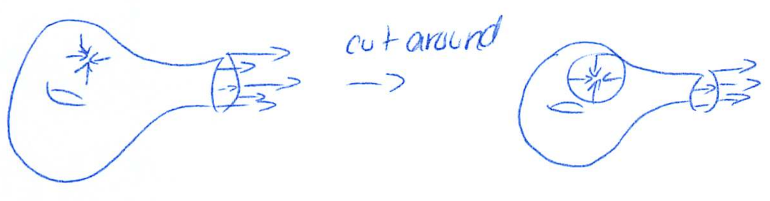


inorder to prove we need a lemma:

$V: x \to \mathbb{R}^n$ a smooth v.f on x.
where x is compact n-manifold with boundary $\partial x = \cup S^{n-1}$ (being a union of spheres). and is V outward pointing along ∂x.

$\Rightarrow \sum_{\text{zeros of } V} i(z) = \deg(g)$ (sum of index of zeros)

$g: \partial x \to S^{n-1}$ (gauss map takes the boundary to the sphere.)



- for each zero z_i of our vector field on X we remove a sphere around it (z_i) to get some new $\tilde{X}$.

- now v has no zeros on $\hat{X}$, can form a new vector

$w = \frac{v}{\|v\|}$ on $\hat{X}$ since every vector of v is non zero

every $|w(x)| = 1$

- on each component of ∂x we have a  circle with a unit vector $g: \partial x \to S^{n-1}$

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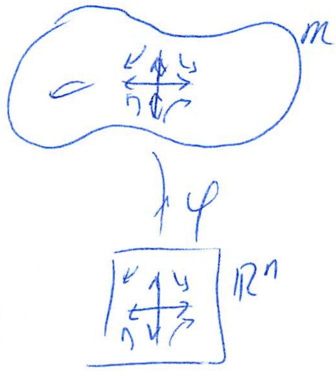
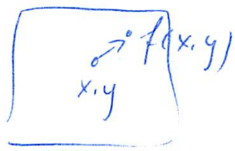
Suppose on a manifold with some zero.



Aside: in calc III

- a vector field on $\mathbb{R}^n$ is just a map $f: \mathbb{R}^n \to \mathbb{R}^n$

Ex. $f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$

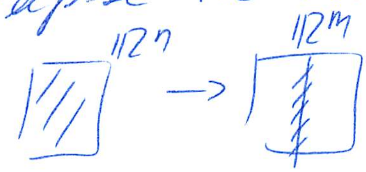
df 

can view as an arrow between two points

- vector fields on $\mathbb{R}^n$ we think of as the derivatives $df: \mathbb{R}^n \to \mathbb{R}^n$ can integrate the vector field df is a linear map.

- if v has an isolated zero then df will be nonsingular matrix.

- $df: \mathbb{R}^n \to \mathbb{R}^n$ and is some linear map and it has a matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}$

if df is singular then it will collapse in some direction: 

↳ if the derivative is a singular mapping means we have a non isolated zero of f. df is less then full rank here.

- if the derivative is nonsingular $\mathbb{R}^n \to \mathbb{R}^n$ means the determinant is nonzero. ($\det df_x \neq 0$)

given $f(x)=0$ $\iff$ f has isolated non degenerate zero at x.

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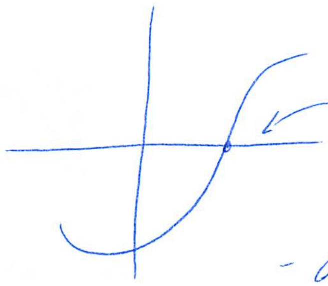
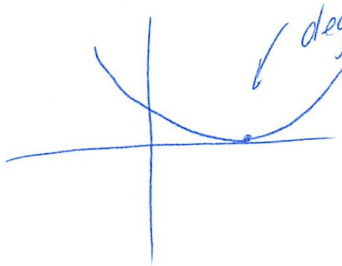
- In the dim $m = 1$



derivative is non zero

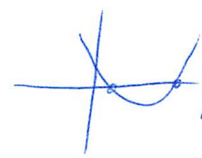
$f(x) = 0 \quad f'(x) \neq 0$

- a move preserves the nature of the zero in this case. Sometimes called transversality.



$f(x) = 0$
$f'(x) = 0$

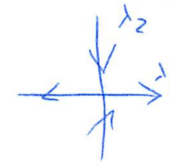
- if we push it a little changes the nature of the zero



could become two or could go away.

Two cases:

1. $\det df < 0 \iff i(x) = -1$ → product of eignvalues is negative (one neg & one pos)



2. $\det df > 0 \iff i(x) = +1$ → product of eignvalues is positive. (Have same sign)

linear map

$$df \begin{pmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \lambda_n \end{pmatrix}$$

Jordan canonical form
- can always make every linear map into an upward tri matrix. has the eignvalues of df along the diagonal. ∴ index = sign of $\det df_x$.

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