CLASS NOTE
14 NOV 2011

Topology.

Vector fields.

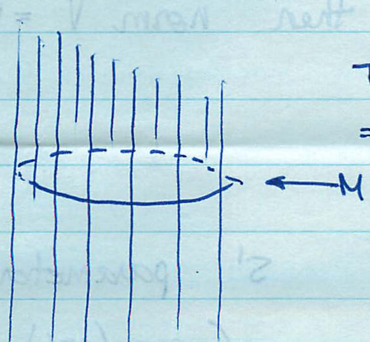
Def: a vector field on a smooth manifold. M is an assignment of an element of TM_x to each point $x \in M$.

$$V : M \to TM$$

~~Tangent bundle~~:

V is smooth, if this assignment is a smooth function.

Example: $M = S^1$



$TM = S^1 \times \mathbb{R}$
$=$ cylinder.

Sometimes $V(x) = 0$. (here x is a singularity or a zero of v)

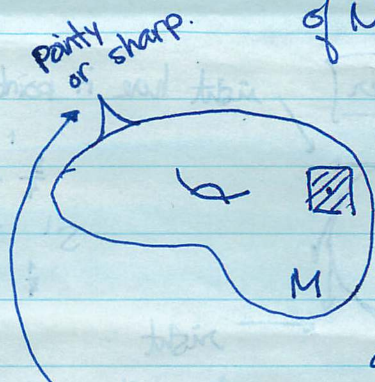
if $V(x) \neq 0$, can get a directional field, by new vector field

$$V'(x) = \frac{V(x)}{\|V(x)\|}.$$

Direction field captures all the topological information.
- Multiplying by constant in a differential equation.
- Rescaling time.

Aside: Recall

TM_x = tangent space of M at x.



(in our case, M of dim n $TM_x \cong \mathbb{R}^n$ at every x

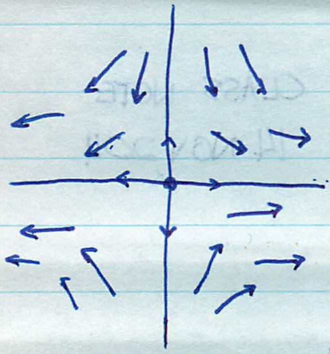
if then not a smooth manifold.

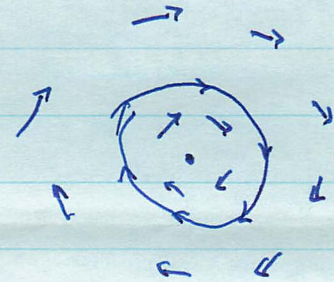
TM = tangent bundle of M.
$\{ TM_x \mid x \in M \}$.

① $t \to (\cos(t), \sin(t))$
$[0, 2\pi) \to S^1$

② $t' \to (\cos(2\pi t), \sin(2\pi t))$
$[0, 1) \to S^1$ (1)

Both ① and ② are the same ! $\cong$ diffeomorph





$\boxed{\text{saddle}}$ $\neq$ $\boxed{\text{center.}}$

can assign a number called the index at x $i(x)$

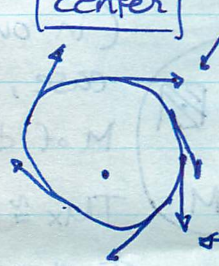
if x is an isolated zero of V, then we can take a small ~~circle~~ sphere C around x containing no singularities.

$$V : \overset{\cong S^{n-1}}{C} \to TM \cong \mathbb{R}^n$$

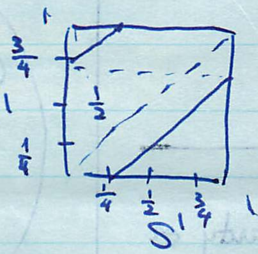
if V is a directional field then norm $V = 1$ $(\|V\| = 1 \text{ on } C)$

so really $V : S^{n-1} \to S^{n-1}$

$\boxed{\text{center}}$ right here it points to 0.

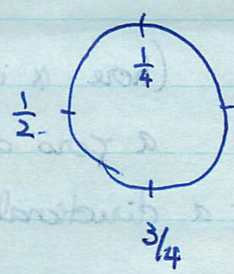


right here, it points to $\frac{3}{4}$.



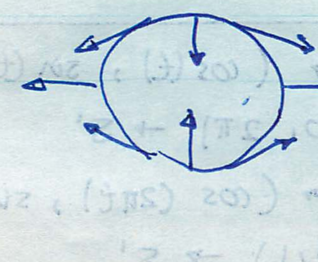
deg = 1

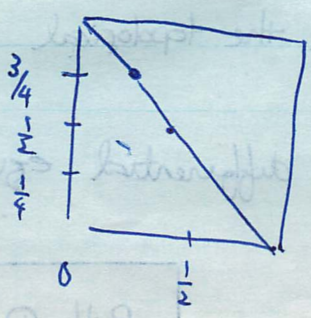
S^1 parameter by $(\cos(2\pi t), \sin(2\pi t))$



i.e. angle in turns.

$\boxed{\text{saddle}}$





deg = −1

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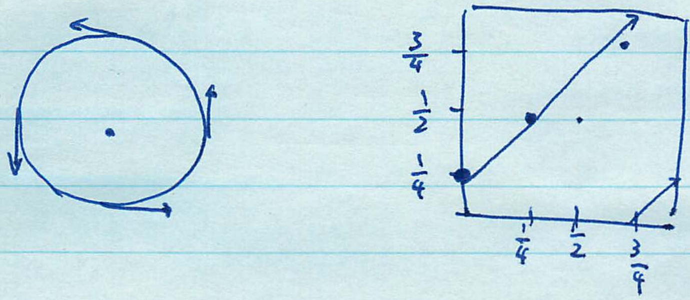
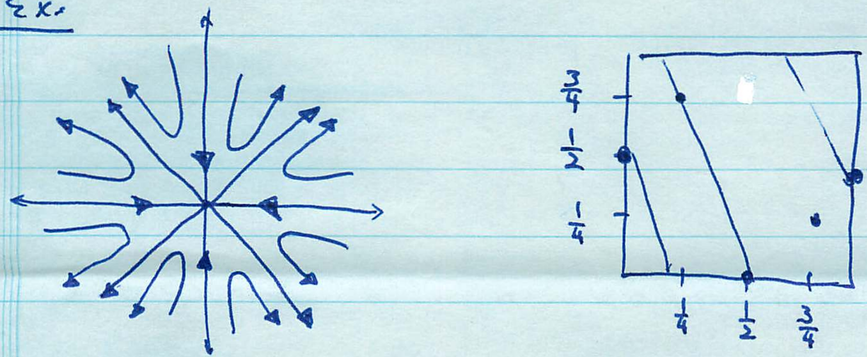
Ex.



deg = 1.

Ex.



deg = −2.

Thm: if V and W are two ~~direction~~ vector field with an isolated singularity (zero) at x and y resp.

$$V(x_0) = 0 = W(y_0)$$

−AND−

there is diffeomorphism f with $f(x) = y$ $[f(x_0) = y_0]$.

−and− $df : V(x) = W(y)$

then. $i(V_x) = i(W_y)$ = degree thing.

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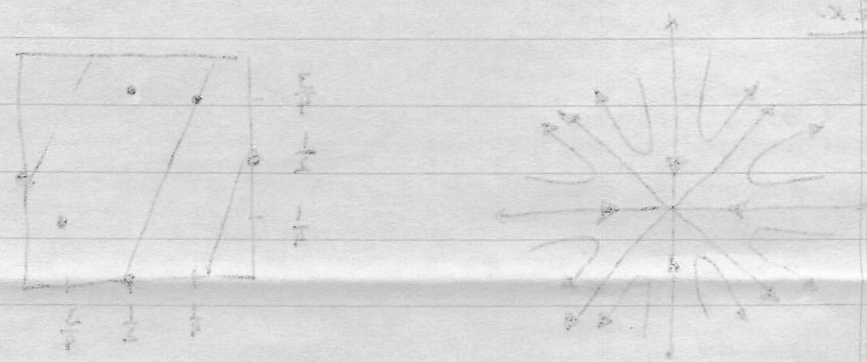
Ex.

deg = 1

Ex.



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$df_x \quad f(x) \rightarrow$

-and- $\quad df : V(x) = W(y)$

then $\quad i(V_x) = i(W_y) = $ degree thing.