11.11

Brower

$$\deg(f) = \sum_{x \in f^{-1}(y)} \operatorname{sgn}(Df_x) \quad y \text{ and REG pt.}$$

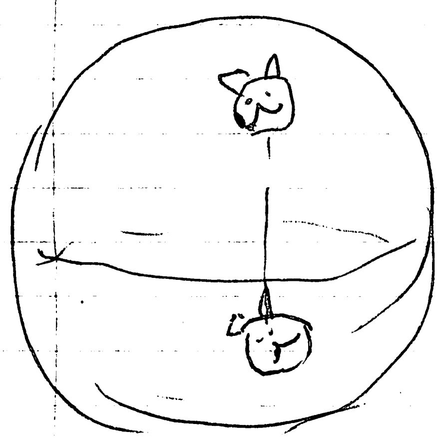
$f \sim g$ homotopic $\quad \deg f = \deg g$

Example. $f \; S^1 \to S^1$

$$f(\theta) = k\theta \quad \deg = k$$

$$f : \mathbb{C} \to \mathbb{C} \quad z \mapsto z^k \quad \deg f = k$$

orientation reversing deffeo of any compact manifold.



$$(x_1 \cdots x_i \cdots x_{n+1}) \in S^n$$

$$\downarrow \text{antix}$$

$$(x_1 \cdots -x_i \cdots x_{n+1})$$

change sign in ith place

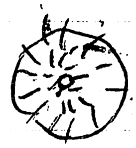
$$D\text{ant} \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & -1 & \\ 0 & & & 1 \end{pmatrix}$$

$$\det = -1$$

"Hairy Ball Theorem."

There is no vector field (with $|v| > 0$ at all points) on S^{n+1} if n is even.

"Comb the hair"



What's a vector field?

in calc 3

$f: \mathbb{R}^n \to \mathbb{R}^n$

gives a vector field on $\mathbb{R}^n$

"attach $F(x) - x$ at x."

? $x \in \mathbb{R}^n$ $F(x)$

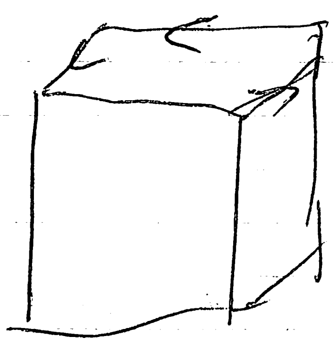
A vector field in $\mathbb{R}^n$ to each point $x \in U$

A vector field (in $U \subseteq \mathbb{R}^n$), to each pt $x \in U$ attach a vector $V(x)$

Vector Field is smooth if $V(x)$ varies smoothly with x.

if vector field on S^n

where does $V(x)$ live? $v_x \in TM_x$

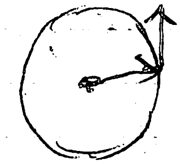


for $x \in S^n \subset \mathbb{R}^{n+1}$

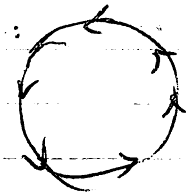
$$S^n = \{(x_1 \cdots x_{n+1}) \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

need $v(x) \perp \vec{x}$

i.e. $v(x) \cdot x = 0$



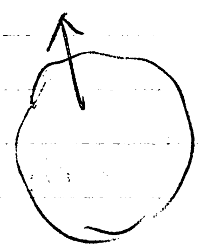
if odd.



want to show that if n is even must have $v(x)$ somewhere.

if $v(x) \neq 0$, can assume $|v(x)| = 1$

(if not, make vector field $\frac{v(x)}{|v(x)|}$ instead)



Suppose I have some vector field with $|v(x)| = 1$ everywhere on S^1

make a homotopy.

$$F: S^n \times [0, 1]^{\ni \theta} \to S^n$$

by

$$\begin{cases} F(x, \theta) = x\cos(\theta\pi) + v(x)\sin(\theta\pi) \\ F(x, 0) = x \\ F(x, \tfrac{1}{2}) = v(x) \quad \leftarrow \text{This would be bad} \\ F(x, 1) = -x \end{cases}$$

for each θ, since $|v(x)| = 1$

$$|x\cos(\theta\pi) + \sin(\theta\pi)v(x)|$$

$|x| = 1$ $\qquad |v(x)| = 1$

so $|F(x, \theta)| = 1$ for all x

since we have homotopy from identity $\swarrow$ deg $= 1$

to identity $\neq$ deg $= -1$

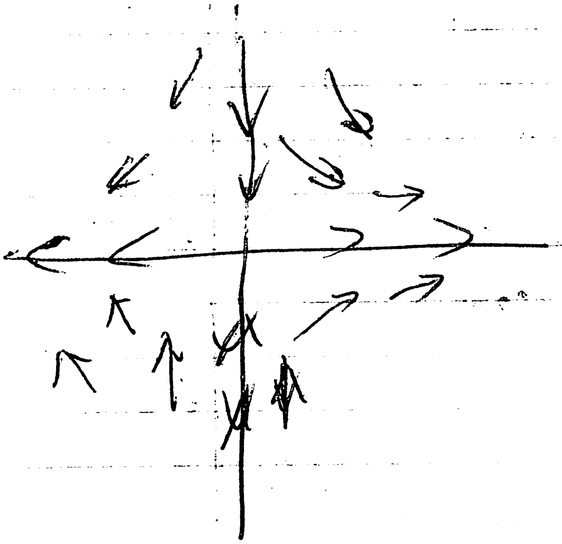
$(x_1 \; - - \; x_{n+1})$

$\downarrow$

$(x_2, -x_1, x_4, -x_3 - \; x_{n+1}, -x_n)$

gives non-zero vector field on odd Dim sphere

next topic is to look at vector field more closely



$\frac{dx}{dt} = F(x)$

$\downarrow$

$\varphi(x)$

$x(0) = x$

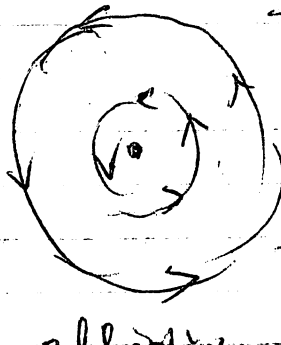
possible sol'ns of D.E's on manifold are related to topology of M

$(\theta, \varphi) \to (\theta + l, \varphi)$

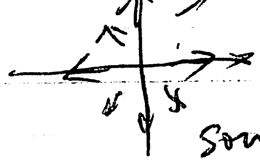
spose I have a vector field with a singularity

(i.e. $V(x) = 0$)

in $\mathbb{R}^2$



elliptic



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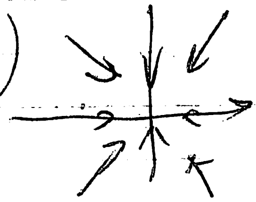
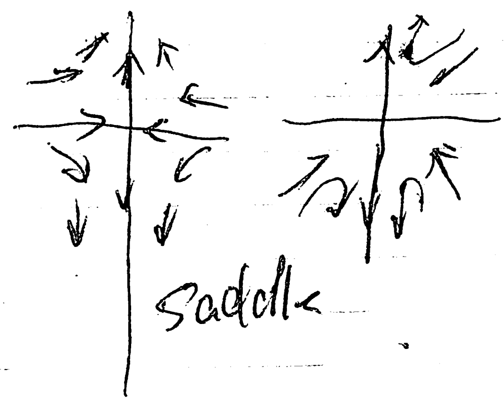


sink



saddle