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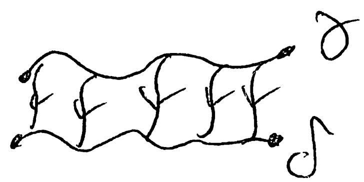
Mat 364: 10/24 notes

Midterm information

- 3 problems. At least 1 will be a students. 1 HW problem (suitably adapted)
- will cover up to Sard's Theorem
- Topics: i) openness, closed, compactness, (path) connected, regular points + values $\# f^{-1}$
  ii) Fundamental Th.m of Algebra, Brouwer F.P. Th.m, Sard's Th.m
  Note: This list is certainly not exhaustive!

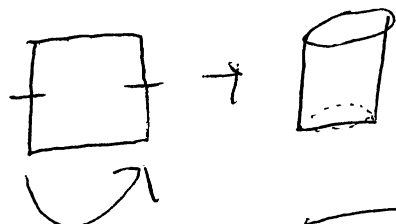
Know techniques of major theorems, ~~don't~~ will not be asked to reproduce

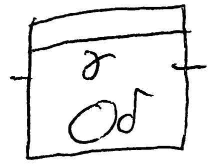
Homotopy

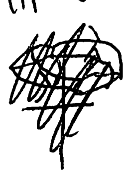
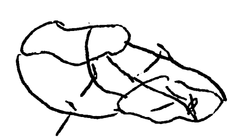
Homotopy of paths 

continuous transformation of one path to another.

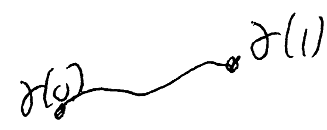
If paths are not homotopic, something is in the way.

Consider  This space is a cylinder.

Look at two paths 

δ is not homotopic to γ. Therefore the space is not $\mathbb{R}^2$
In $\mathbb{R}^2$ all closed paths are homotopic ~~to each~~ to each other.
 In $\mathbb{R}^2$ simple translation will do. 

Homotopy of paths gives information on the underlying topological structure.

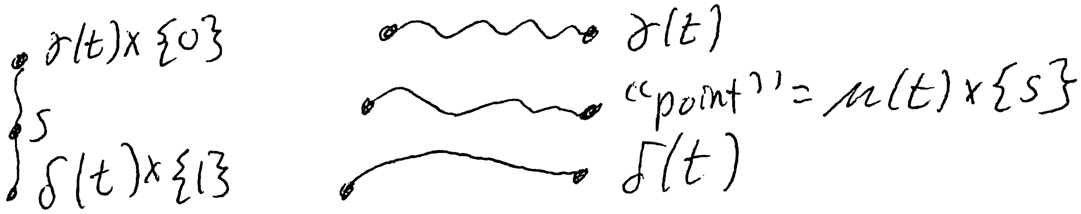
A path γ in a space X is a map $\gamma : [0,1] \to X$ 

If ~~___~~ $\gamma(t) \sim \delta(t)$ the (smooth) homotopy is a pairing of $\gamma(t_0)$ with a $\delta(t_0)$ for each t_0 that starts with $\gamma(t_0)$ and ends at $\delta(t_0)$ which is a path, varying continuously.

It's a path in the space of paths.

$\gamma(t)$ is a point in the space of paths: $X \times [0,1]$

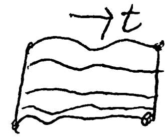
$\delta(t)$ is another point in ~~___~~ $X \times [0,1]$



Aside: The trefoil knot  is not homotopic to the unknot  in $\mathbb{R}^3$, but is in $\mathbb{R}^4$

More formally: For $Y \subset \mathbb{R}^n$, paths $\gamma(t)$ and $\delta(t)$ (maps from $[0,1] \to Y$) are smoothly homotopic if there exists a map $F(t,s)$ such that $F: \underset{\gamma + \delta}{\overset{\uparrow}{[0,1]}} \times \underset{s}{\overset{\uparrow}{[0,1]}} \to Y$ with $F(t,0) = \gamma(t)$, $F(t,1) = \delta(t)$, and F smooth

$F(t, s_0)$ corresponds to a path in between where s_0 is a fixed value and t is allowed to vary.



Can generalize this further.

Homotopy of functions:

Def'n: $f: X \to Y$, $g: X \to Y$ are (smoothly) homotopic if there exists a function $F: X \times [0,1] \to Y$ such that $F(x,0) = f(x)$, $F(x,1) = g(x)$ and F is smooth.

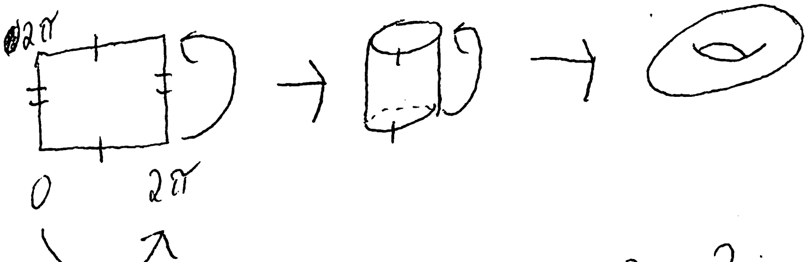
Def'n: f is smoothly isotopic to g if f is homotopic to g (i.e. there is a homotopy F s.t. $F(x,0) = f(x)$, $F(x,1) = g(x)$, & F is smooth) and for the homotopy F the map $x \to F(x,s)$ is a diffeomorphism of $X \to Y$ (for all s)

At every s there is a function $X \to Y$

Ex) $f: T^2 \to T^2$ where T^2 is the 2-dimensional torus $S^1 \times S^1$

(note: In class the notation was $\mathbb{T}^2$ but this isn't an upper case pi:)

$T^2 = \{(\theta, \varphi) \mid 0 \le \theta, \varphi < 2\pi\}$ with $(\theta, 0) \sim (\theta, 2\pi)$ ie these points
$(0, \varphi) \sim (2\pi, \varphi)$ are identified.



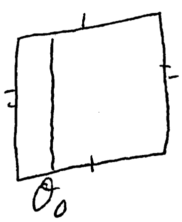
Have a second map $g: T^2 \to T^2$.

$f(\theta, \varphi) = (\theta, \varphi)$

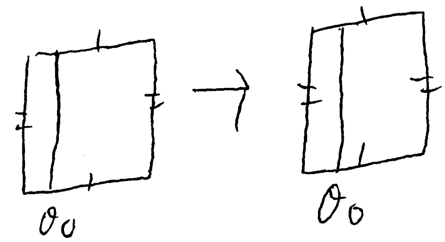
$g(\theta, \varphi) = \left(\theta, \varphi + \left(\frac{\varphi - \pi}{2\pi}\right)^2\right)$

$g \sim f$ (isotopic too.)

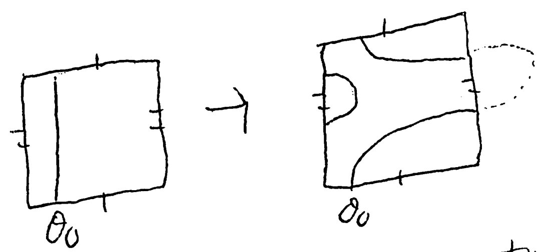
Consider a line of constant θ



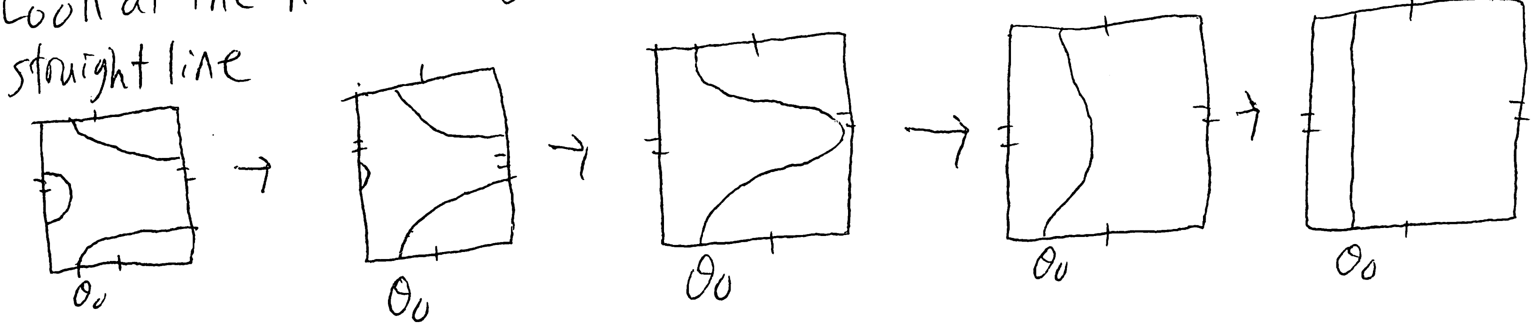
under f, this line is unchanged f:



For g:



Look at the ~~new~~ line g maps to. It can be straightened out to a straight line



We can construct an explicit homotopy

$$F(\theta,\varphi,s) = \left(\theta, \varphi + s\left(\frac{\varphi-\pi}{2\pi}\right)^2\right)$$

$$F(\theta,\varphi,0) = f(\theta,\varphi)$$

$$F(\theta,\varphi,1) = g(\theta,\varphi)$$

Degree

Want to generalize the idea of degree

why?

a) It's fun b) useful in topology c) want to define degree of f.

Consider $h(\theta,\varphi) \to (\theta, 2\varphi \bmod 2\pi)$

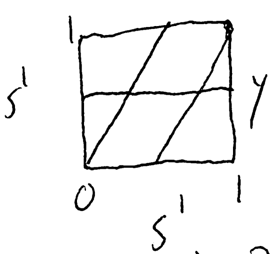
degree of f is 1. degree of h is 2 h is not homotopic to f

Model is $\mathbb{C} \to \mathbb{C}$

Polynomial of degree d wraps $\mathbb{C}$ around itself d times.

$$p(z) = \sum_{i=0}^{d} a_i z^i$$

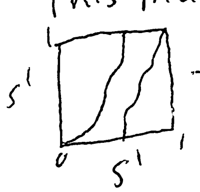
If y is regular, $\# p^{-1}(y) = d$



$$f(\theta) = 2\theta \bmod 1$$

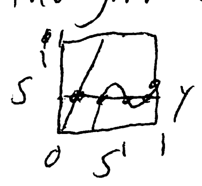
$\# f^{-1}(y) = 2$

This map is of degree 2.



$\to$ This map is homotopic to f. Should also have degree 2.

Problem though: homotopy doesn't imply the degrees are equal.

Consider  at y this map has degree 4 and is homotopic to f, but $\# f^{-1}(y) = 2$

If $f \sim h$ then $\deg(f) = \deg(h) \bmod 2$.