MAT 364 Notes

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We're proving Sard's Theorem!

Sard's Theorem 1. *If $U \subseteq \mathbb{R}^n$ is open and $f : U \to \mathbb{R}^p$ is smooth, with $\mathcal{C}$ as the critical points for the function f, then the image $f(\mathcal{C})$ has measure zero.*

Let $\mathcal{C}$ =critical points and let $\mathcal{C}_k = \{x \in \mathcal{C} | partialsvanishtokthorder\}$. We prove this using three steps:

Step 1 $f(\mathcal{C} - \mathcal{C}_1)$ has measure zero. We proved this painstakingly last class.

Step 2 $f(\mathcal{C}_k - \mathcal{C}_{k+1})$ has measure zero.

Step 3 $f(\mathcal{C}_j)$ has measure zero for "large enough" j.

We're now gonna prove step 2! Let

$$w(x) = \frac{\partial^k f}{\partial_{x_{n_1}} \dots \partial_{x_{n_k}}}(x)$$

so that $w(\bar{x}) = 0$ but $\left.\frac{\partial w}{\partial x_1}\right|_{\bar{x}} \neq 0$. We can choose the coordinates without loss of generality so that this is so.

We now create $h : U \to \mathbb{R}^n$ where $h(x) = (w(x_1), x_2, x_3, \dots, x_n)$ so h is a diffeo of neighborhood $V_{\bar{x}}$ to neighborhood V'. Note that $h(\bar(x)) = (0, \bar{x}_2, \bar{x}_3, \dots, \bar{x}_n)$. We want to claim that $h(\mathcal{C}_k \cap V) \subseteq \{0\} \times \mathbb{R}^{n-1}$ because h makes stuff around $h(\bar{x})$ straighter, since $\left.\frac{\partial w}{\partial x_1}\right|_{\bar{x}} \neq 0$.

To give an example: let $f(x) = (x-1)^5$. Then we see that $f^{(4)}(1) = 0$ but $f^{(5)}(1) = 5!$. In this case, $w = f^{(4)}$ and $\bar{x} = 1$, because that's like the same as our definition. The "straightness" made just means that it has a constant slope around its neighborhood.

Right, back to the proof! We can straighten it our one dimension less. Let $g = f \circ h^{-1}, g : \mathbb{R}^n \to \mathbb{R}^p$, or $g : \mathbb{R}^1 \times \mathbb{R}^{n-1} \to \mathbb{R}^p$. Let $\bar{g} : \{0\} \times \mathbb{R}^{n-1} \to \mathbb{R}^p$, which is g restricted to that $x_1 = 0$ slice of V', which is that straight-line neighborhood of $h(\bar{x})$. By induction, the measure of the critical values of $\bar{g}$ is 0. (Since partials $\leq k$th order have measure 0).

Keep reducing down dimensions. $\bar{g} \circ h((\mathcal{C}_k - \mathcal{C}_{k-1}) \cap V) = f((\mathcal{C}_k - \mathcal{C}_{k-1} \cap V)$. As before, we can pick a countable number of Vs to make up $\mathcal{C} - \mathcal{C}_{k-1}$, so the critical values have measure 0.

Onto step 3! Let $I^n \subseteq \mathbb{R}^n$ be a cube of side length δ. We want to show that, for k large, $f(\mathcal{C}_k \cap I^n)$ has measure 0. This is the heart of the artichoke, after pealing all the layers away inductively.

Forgive me if this gets confusing, but we're gonna go into analysis a wee bit. Note that $k > \frac{n}{p-1}$.

We have $f : U \to \mathbb{R}^p$ being smooth, and I^n is compact, so we can apply Taylor's Theorem: $f(x+h) = f(x) + R(x,h)$, where R is the remainder. $\|R(x,h)\| < c \cdot \|h\|^{k+1}$ where c is the max $(k+1)$th derivative of f on I^n. Basically, this says that, when h is small, $f(x+h) = f(x)$ plus a really really tiny part, so f doesn't move a whole lot.

We're basically done, but we're gonna do more analysis anyway. Chop up I^n into r^n cubes of side δ/r. Any point in I_1, one of our little cubes, is $x+h$ with $\|h\| < \sqrt{n}\frac{\delta}{r}$. This just means the points in the little cube are close together.

$f(I_1)$ isn't distorted (which is what Taylor's Theorem says) and is no bigger than $\frac{a}{r^{k+1}}$ where $a = 2c(\sqrt{n}\delta)^{k+1}$.

WHATEVER! The point is that f can't squish up our little cube too much. That means $f(\mathcal{C}_k \cap I^n)$ isn't too bad either. The sum of the cubes isn't too stretched. Its r^n cubes, with total volume $< r^n\left(\frac{a}{r^{k+1}}\right)^p = a^p r^{k-(k+1)p}$. Now, let k be large, and this gets pretty small as r goes to ∞.

Sorry there's no pictures. If you want a picture, look at figure 5 on page 17 of Milnor. The full proof of Sard's Theorem is Section Three of Milnor. Don't be discouraged if you didn't get it the first time; I still don't totally get it!