NOTES FOR MAT 364 14 OCT 2011
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SARD'S THM

Let $U \subseteq R^n$ open, $f : U \to R^P$ smooth

C=$\{x|rank\ Df_x < p\}$=critical points of f, then $f(c)$ has measure 0 in R^P.

The case where n$\leq p$ is easy, but we'll do it hard way anyway.

PROOF/ By induction on n.

If n=0, then U is a point, so trivial.

Proof in 3 steps

Divide C into subsets by how many derivatives vanish.

Example:

In n=1, p=1

$f(x) = x^4(x-1)^3(x-2)^2$

$C = \{0,1,2\}$

$\{0,1,2\}$=C_1=points where $f' = 0$.

$\{0,1\}$=C_2=points where $f' = 0, f'' = 0$.

$\{0\}$=C_3=points where $f' = 0, f'' = 0, f''' = 0$

$$C \supset C_1 \supset C_2 \supset C_3 \supset \cdots$$

$$C_1 = \{x|Df_x = 0\}$$

$$C_k = \{x|all\ partials\ to\ order\ k\ = 0\}$$

STEP 1: Show that $f(C - C_1)$ has measure 0;

STEP 2: Show that $f(C_i - C_{i+1})$ has measure 0;

STEP 3: Show that $meas(f(C_k))$= 0 for k large.

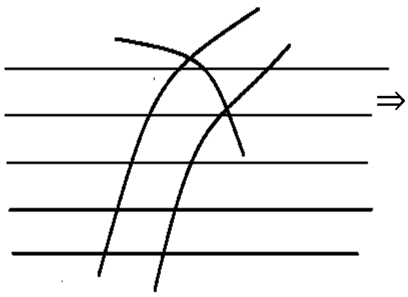
1. Going from $C_1 \to C$ doesn't add measure to critical values;
2. $C_k \to C_{k+1}$ doesn't lose measure to critical values;
3. For k big (k>n/p-1), meas $(f(C_K)) = 0$.

Colloquial SART THM: If f wiggles too much, it ain't smooth.

PF/ assume p≥2 (since if p=1, $C = C_1$)

Rely on Fubini

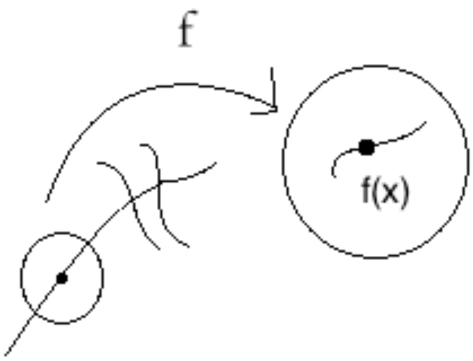
If $A \subset R^P$ has measure 0, if on every hyperplane $\{C\} \times R^{P-1}$, the intersection $A \cap \{C\} \times R^{p-1}$ has p-1 dim measure 0.



⇒ horizontal cuts have measure 0,

whole set has p-dim measure 0.

For each point $\bar{x}$ in $C - C_1$, we want to find $V \subset R^n$ open, $\bar{x} \in V$, so that $f(V \cap C)$ has measure 0, countably many such V does the trick.



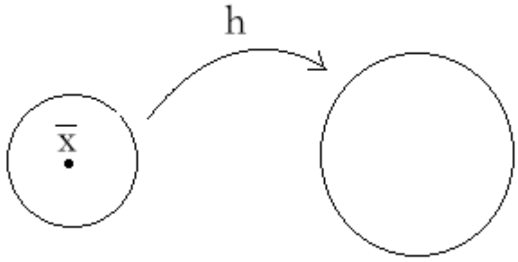
Since $\bar{x} \in C - C_1$, $\frac{\partial f_j}{\partial x_i} \neq 0 \; at \; \bar{x}$.

WLOG: $\frac{\partial f_1}{\partial x_1} \neq 0\ at\ \bar{x}$. (It doesn't have to be x_1, but we can choose x_1.)

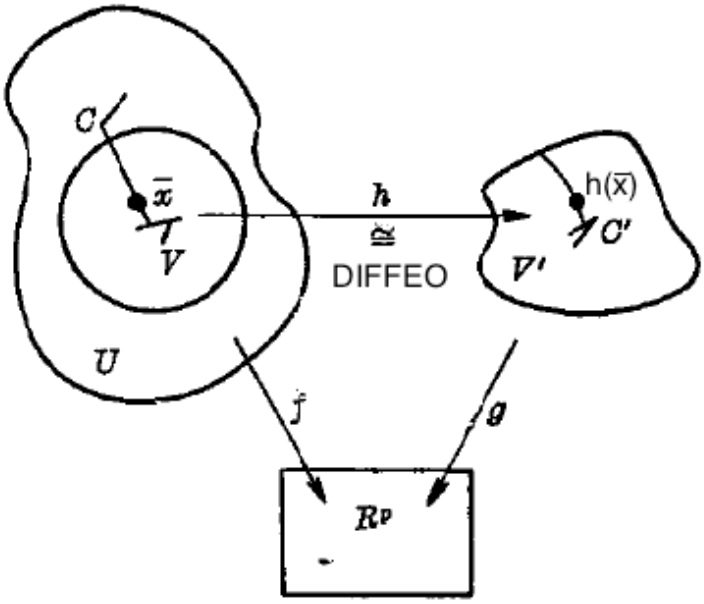
Let h: U→ R^n.

$$h(x) = (f_1(x_1), x_2, x_3, \cdots, x_n), \quad \text{so } dh_{\bar{x}} \text{ is nonsingular.}$$

h: $V_{\bar{X}} \to V'$



$\bar{x}$ *near* C
or in C
or whatever



Take the whole neighborhood onto the neighborhood.