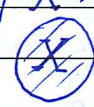
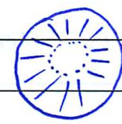
MAT 364 | Oct. 10, 2011

Recall: Lemma: Let X be compact manifold with boundary ∂X, there is no smooth map $f: X \to \partial X$ so that $f(y) = y$ for all $y \in \partial X$



[intuition: must "rip a hole" in X to do this]

Ex:



$\leftarrow \{(x,y) \mid \frac{1}{2} < x^2 + y^2 \leq 1\}$

[This one has such f, but it's not <u>compact</u>.]

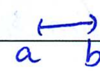
$\boxed{\text{II}} \Rightarrow$ **Boundary:**

$\partial M \neq$ differential of M.

- One argument for using ∂M = boundary of M is via Stokes' thm.

  F.T.C. (fundamental theorem of calculus) states

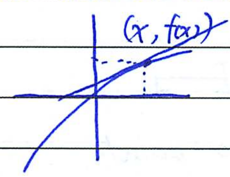
  $$\int_a^b \frac{\partial f}{\partial x} dx = f(b) - f(a) = f \text{ on } \partial I \text{ (interval)}$$



$\leftarrow$ 1-dim

$\leftarrow$ high-dim.

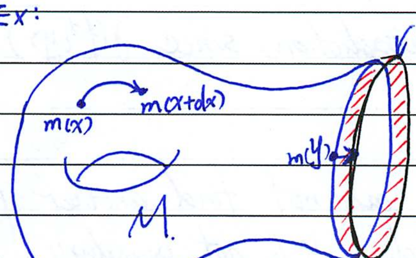
- What is a derivative?



if $x \to x + dx$

then $f(x) \to f(x) + f'(x)dx$

Ex:

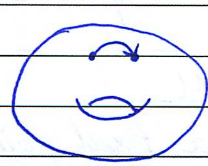


$\leftarrow$ manifold with boundary

the points in the boundary change

"increase M a little"

Ex:



$\leftarrow$ compact manifold. $\partial(\circledcirc) = \emptyset$.

The points don't leave the manifold.

$\boxed{\triangleright} \Rightarrow$ **Proof:** the Lemma. by Apply previous stuff.

Suppose f is such a map. Let $y \in \partial X$.

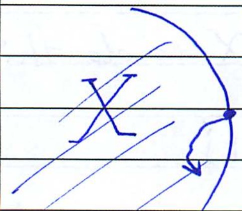
(we know) $Id|_{\partial X} = f|_{\partial X} : \partial X \to \partial X$ is the identity

so y is a regular value for $f|_{\partial X} = Id|_{\partial X}$.

$f^{-1}|_{\partial X}(y)$ is a smooth submanifold of ∂X.

$f^{-1}(y)$ is a smooth manifold with boundary.

$f^{-1}(y) \cap \partial X = f^{-1}|_{\partial X}(y)$



y ← This picture is wrong.

Pf: X is n-dim

∂X is $n-1$ dim

$f: X \to \partial X$

$Id = f|_{\partial X}: \partial X \to \partial X$

0-dim $\quad (f|_{\partial X})^{-1}(y) = \{y\}$

singular point $\{y\} = (f^{-1}(y) \cap \partial X)$ is 1-dim smooth with boundary.

$\exists$ only 2 kids 1-dim smooth manifold. ⇐ (proof next time)

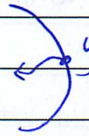
$\longleftrightarrow \quad \bigcirc$

& 1 with boundary $[\!-\!-\!]$

$\partial([\!-\!-\!]) = \{a, b\}$ two points.

$\quad a \quad b$

but in the picture contradiction, since $\partial(f^{-1}(y)) \cap \partial X$ is 1 point, not 2 or 0.



y ← fix in y, can not find another point.

and if circle, y is not boundary!

COR: $Id: S^{n-1} \to S^{n-1}$ cannot be extended smoothly to $\overline{\mathbb{D}}^n \to S^{n-1}$.

SMOOTH BROUWER FIXED-POINT THEOREM

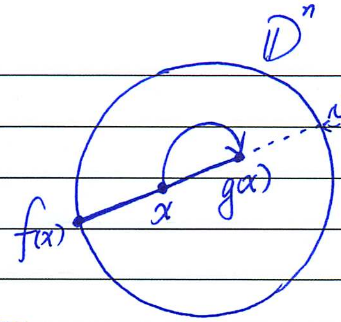
$g: \overline{\mathbb{D}}^n \to \overline{\mathbb{D}}^n$ with g smooth has a fixed point.

Pf: (by contradiction)

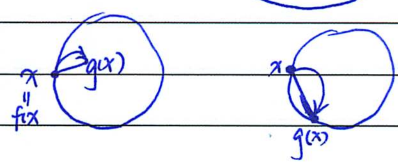
Suppose I have such a g.

Let X be any point in $\mathbb{D}^n$

Define $f(x)$ to be the point on the boundary $\partial\mathbb{D}^n$, and "on the other side" of $g(x)$



no • f is a smooth map from $\overline{\mathbb{D}}^n \to S^{n-1}$
• If $x \in \partial\overline{\mathbb{D}}^n$, $f(x) = x$.
Now have contradiction $f(x) = x \Rightarrow \in$



[open to open] can not have a fixed point.

probably the fixed point on $\partial\mathbb{D}^n$ doesn't compact, then doesn't contradict.

Brouwer fixed-point theorem:
No homeomorphism $f: \overline{\mathbb{D}}^n \to \overline{\mathbb{D}}^n$ which is fixed.
"pf" (idea)
- extend smooth → no smooth.
- [weierstrass function] (it continous every where, but differentiable nowhere)
- Any homeomorphism of $\mathbb{R}^n \to \mathbb{R}^n$ can be smooth-approximat by a polynomial function.