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Mat 364 Notes

Fundamental Theorem of Algebra:

If $P(Z)$ is a polynomial, $P: \mathbb{C} \to \mathbb{C}$ (of degree d), then $P(Z)=0$ has a solution.

(In fact, it has d solutions with multiplicity).

Recall what $\mathbb{C}$ is,

$\mathbb{C} \to \mathbb{R}^2$.

$\mathbb{C} \to Z = a+ib \leftrightarrow (a,b) \in \mathbb{R}^2$. vector space

$(a+ib)(c+id) = (ac-bd) + i(ad+bc)$.

$(a+ib)+(c+id) = (a+c) + i(b+d)$.

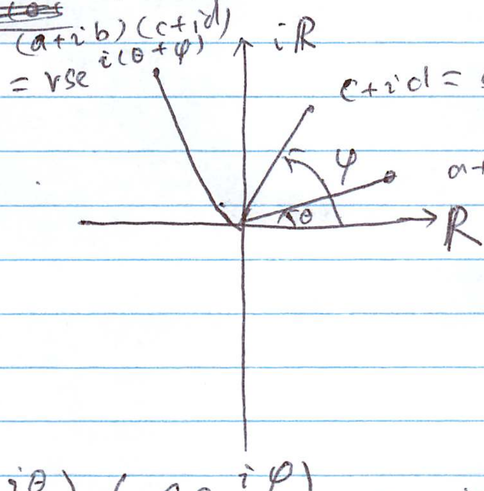
Add point at ∞ to $\mathbb{C}$ to get

$\mathbb{C} \cup \{\infty\} = \bar{\mathbb{C}} \cong$  $\to$ Riemann sphere (spherical projection)

~~(a+ib)(c+id)~~

~~= rse i(θ...~~

$\mathbb{C}$



(polar coordinates (r,θ))

$$(re^{i\theta})(se^{i\varphi}) = re^{i(\theta+\varphi)}$$

$Z \to kZ$ in $\mathbb{C}$,

$(x+iy) \to (a+ib)(x+iy)$.

$(x,y) \to (ax-by,\ bx+ay)$.

What is this maps as $\mathbb{R}^2 \to \mathbb{R}^2$?

$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} ax-by \\ bx+ay \end{pmatrix}$ i.e. $\begin{pmatrix} a & -b \\ b & -a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

so, dm is $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

columns are orthogonal, $\begin{pmatrix} a \\ b \end{pmatrix} \perp \begin{pmatrix} -b \\ a \end{pmatrix}$

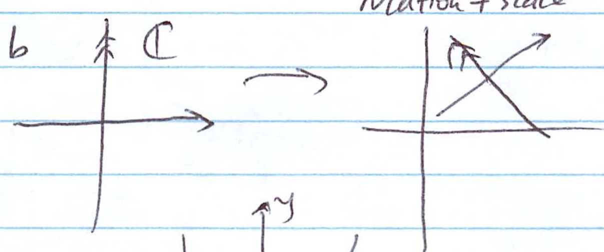
i.e, maps is rotation + scale factor.

Polynomials:

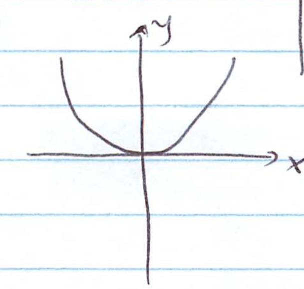
$$Z \mapsto \underbrace{a_d Z^d + a_{d-1} Z^{d-1} + \cdots + a_1 Z}_{} + a_0 = \sum_{i=0}^{d} a_i z^i$$

$a_1 Z \downarrow$ rotation + scale $\qquad a_0 \downarrow$ translation

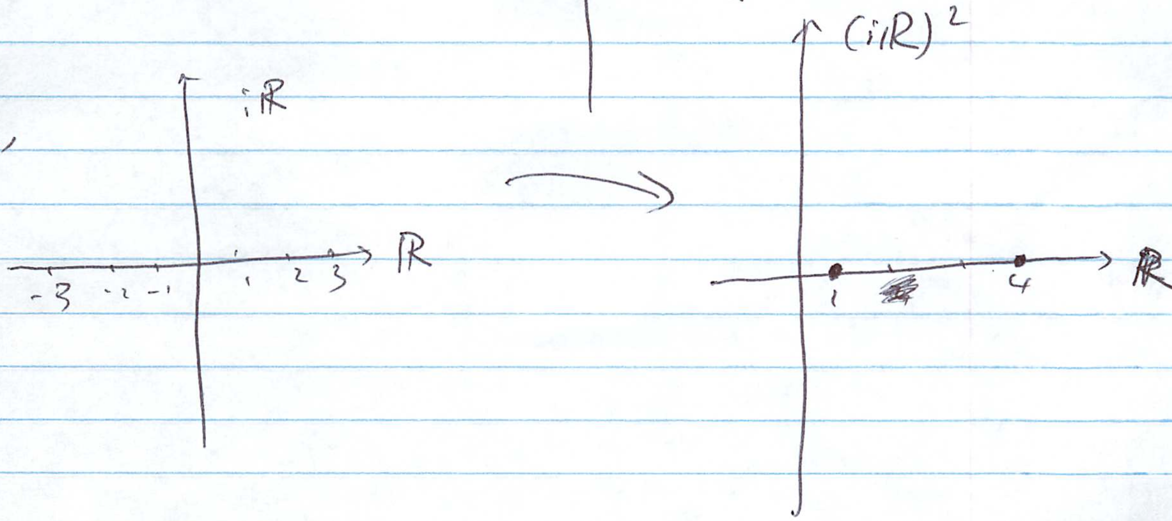
i.e. $Z \longmapsto az+b$ is just

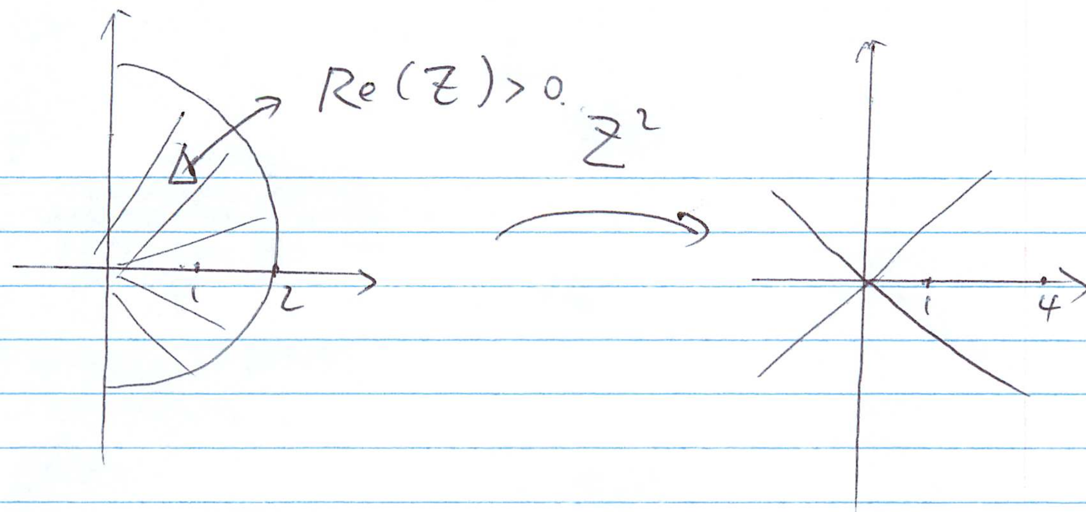


$Z \mapsto z^2$ in $\mathbb{R}$ is



in $\mathbb{C}$,

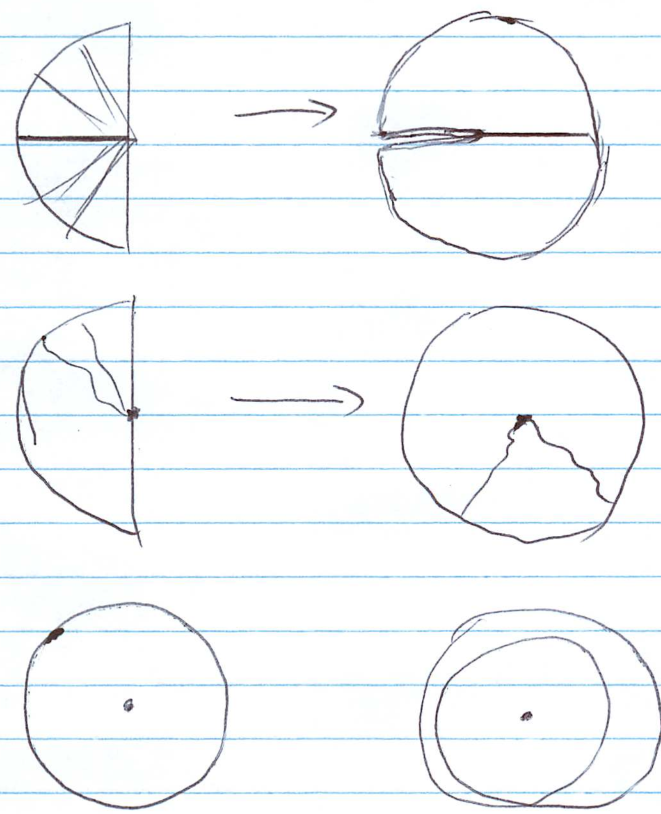




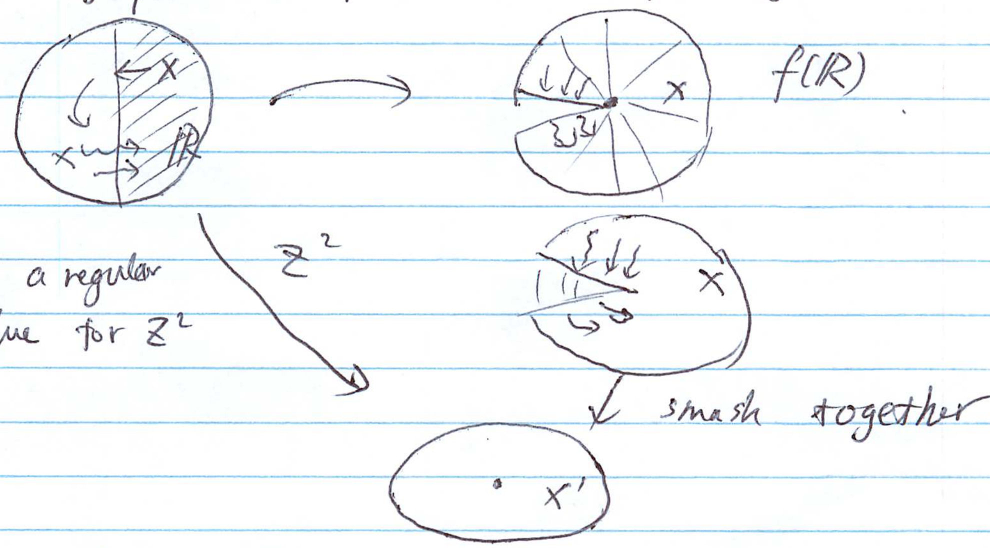
$f(z) = z^2$ takes

$\{z \mid Re(z)>0\} \to \mathbb{C} - \{$negative axis$\}$.

Same for $Re(z)<0$,



For z^2, every point except 0 has 2 preimages.



0 is not a regular value for z^2

Ex. $f(z) = z^2$, what is df?

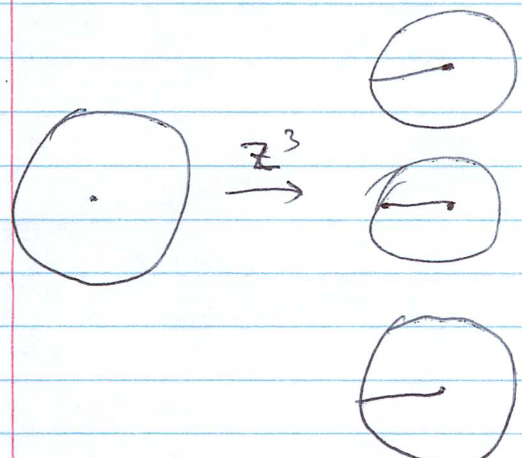
Solution. As map from $\mathbb{R}^2 \to \mathbb{R}^2$,

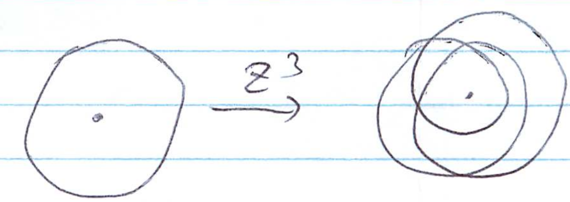
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x & -y \\ -y & x \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}.$$

$$df \text{ is } \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix} = 2 \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

so at any $z \neq 0$, map is nonsingular,
but $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ at 0.

In general, $\frac{d}{dz} z^n = n z^{n-1}$ (only singular at $z = 0$).
($z \neq 0$ is a regular point)



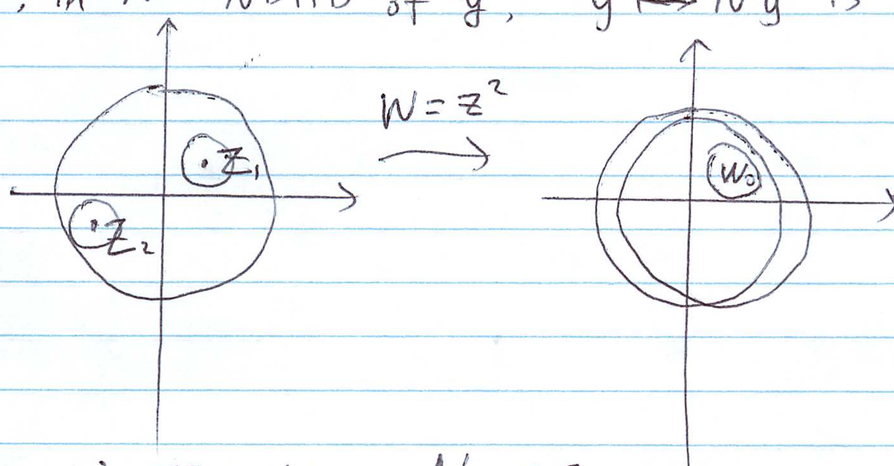


If f is a diffeomorphism with y a regular value,
[i.e. $f(x) = y$, df_x nonsingular].
Define $N_y = \# f^{-1}(y)$.
to be the cardinality of $f^{-1}(y)$.

Ex.

For z^4, $N_1 = 4$, since
$1^4 = 1$, $(-1)^4 = 1$, $i^4 = 1$, $(-i)^4 = 1$,
So # of $f^{-1}(1) = 4$
i.e. $f^{-1}(1) = \{1, -1, i, -i\}$.

Then the function
$y \mapsto N_y$ is constant.
(i.e., in A NBHD of y, $y \mapsto N_y$ is constant).



for w near w_0, $N_w = 2$.

If the disk around w_0 includes 0,
N_w is not 2, since v_0 is a point.