NUJHAT MEHREEN

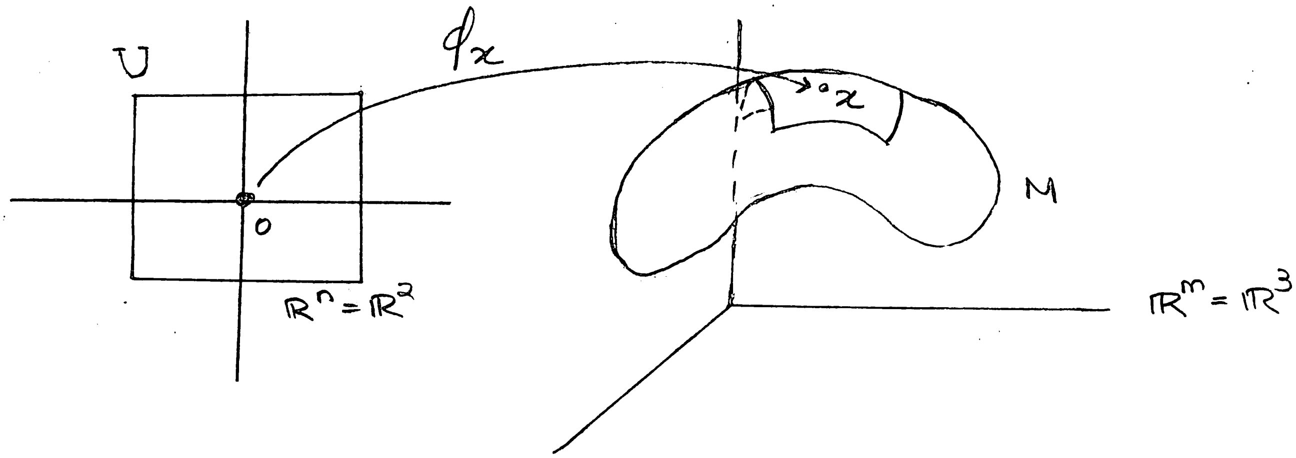
MAT 364

Class - 7

9/19/2011

Consider a smooth manifold $M \subseteq \mathbb{R}^m$. This means that at each $x \in M$, $\exists\, U \subset \mathbb{R}^n$ with a diffeomorphism $\phi_x : U \to \mathbb{R}^m$. [i.e. at each $x \in M$, there is a neighborhood in $\mathbb{R}^n$, U, with a diffeomorphic ϕ_x that takes U to $\mathbb{R}^m$]

Eg: Say, $U \subset \mathbb{R}^2$ and $M \subset \mathbb{R}^3$. For convenience, let the point in consideration in U be $(0,0)$.



$\therefore$ Here, $\phi_x(0) = x$

Since M is smooth, ϕ_x is differentiable at x and has a derivative matrix.

i.e. $D\phi_x$ exists and is invertible (or non-singular).

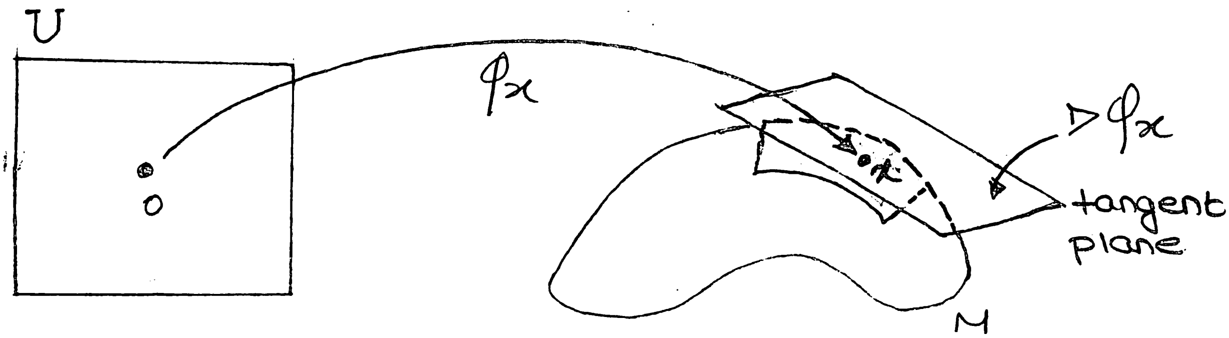
ϕ is the local parameterization here.

$\therefore$ For all x: $x_1, x_2, \ldots, x_n$

$$\phi\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} \phi_1(x_1 \cdots x_n) \\ \phi_2(x_1 \cdots x_n) \\ \vdots \\ \phi_m(x_1 \cdots x_n) \end{bmatrix}$$

$$\Rightarrow \quad D\phi_x = \left.\begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \vdots & & \vdots & \vdots \\ \frac{\partial \phi_m}{\partial x_1} & \cdots & \cdots & \frac{\partial \phi_m}{\partial x_n} \end{bmatrix}\right|_x$$

The derivative $D\phi_x$ is a linear map $\mathbb{R}^n \to \mathbb{R}^m$, $m \geq n$, and is the tangent plane at the point x on the manifold M. It is a plane that simply sits at the point x on M.

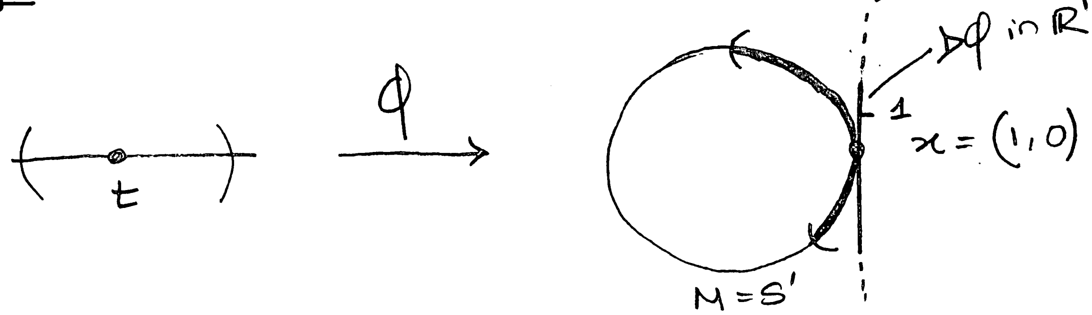


__Definition__: The tangent space at x of M is the image of $D\phi_x : \mathbb{R}^n \to \mathbb{R}^m$

__Notation__: $TM_x \leftarrow$ tangent space to M at the point x.

However, since the tangent space itself has not been defined yet, to check whether this relation is right, it can be shown that TM_x does __not__ depend on the choice of ϕ.

__Eg__: Consider a circle as a manifold,



Here, ϕ takes t to any point x on M.

A parameterization of the circle can be

$$\phi(t) = (\cos(t), \sin(t))$$

$$\Rightarrow D\phi(t) = (-\sin(t), \cos(t))$$

At a given point $x = (1,0)$,

$$D\phi = (0,1)$$

$\therefore$ A more general way of expressing this: $D\phi_x(t) = (0, t)$

$D\phi = (0,1)$ is the linear map from $\mathbb{R}^1 \to \mathbb{R}^2$ and is shown by the vertical line tangent to the circle at $x = (1,0)$

Now, if there is some other parameterization of a given circle, ψ, such that

$$\psi(s) = \left(\sin(2\pi s),\ \cos(2\pi s)\right)$$

Then,

$$D\psi(s) = \left(2\pi \cos(2\pi s),\ -2\pi \sin(2\pi s)\right)$$

Here, $\psi(1/4) = (1,0)$

and at $s = \frac{1}{4}$, $D\psi = (0, -2\pi)$

Therefore,

$$\text{at } x = (1,0),\quad D\phi = (0,1)$$
$$\text{\& at } s = 1/4,\quad D\psi = (0,-2\pi)$$

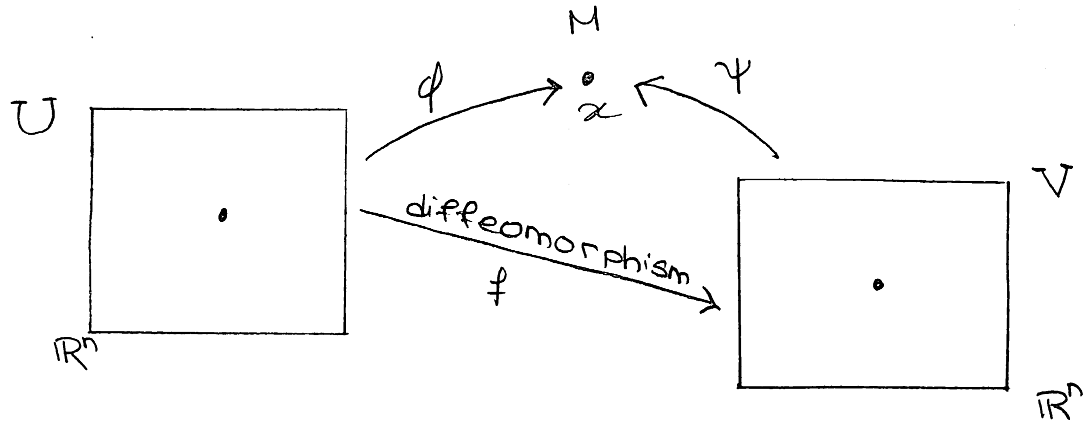
At first look, it seems like these are two different maps. However, they are basically the same identical lines, except for the fact that they are going in opposite directions at different speeds. They are both in $\mathbb{R}^1$ but they differ by a constant, -2π.

i.e. $-2\pi(D\phi_x) = D\psi_x$

or, expressed in general,

$D\phi_x = A D\psi_x$, A is some linear map.

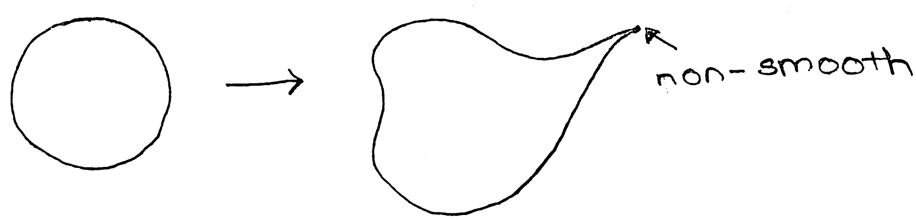
Proof by diagram:



In the tangent space, TM_x

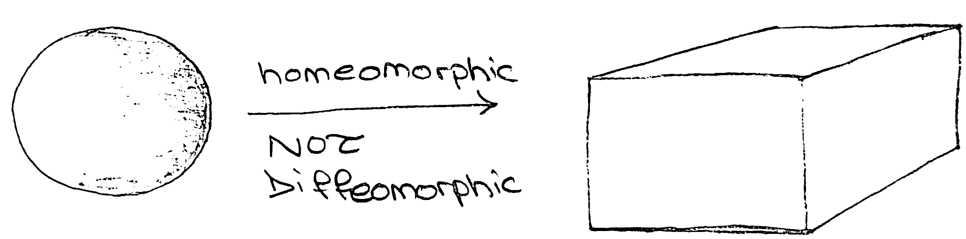
$$\begin{tikzcd}
 & TM_x & \\
\mathbb{R}^n \arrow[ur, "d\phi"] \arrow[rr, "d\phi(d\psi)^{-1}"'] & & \mathbb{R}^n \arrow[ul, "d\psi"']
\end{tikzcd}$$

What was being discussed so far was under the condition that it was a smooth manifold.
Say, now, there is a non-smooth manifold.

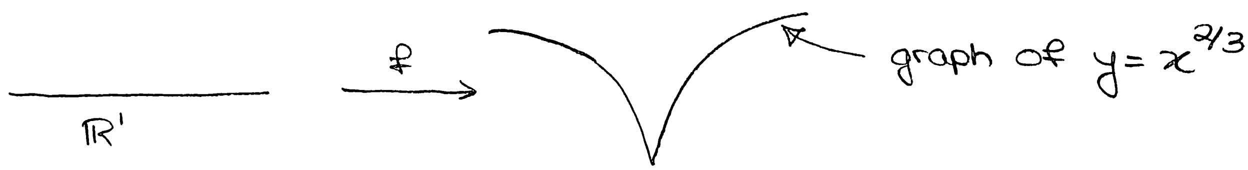


This is still homeomorphic, but the pointy end proposes a problem.

Eg: A sphere is homeomorphic to a cube, since there are edges and corners in the cube making it non-smooth.
$\therefore$ Not diffeomorphic.



Again, consider the example below.



There is no derivative for the graph of $y = x^{2/3}$ at $x = 0$

1 parameteriz ion of this, $f(t) = (t^3, t^2)$

$\therefore \quad Df = (3t^2, 2t)$

At $t = 0$, the matrix becomes $(0, 0)$

This is, therefore, <u>NOT</u> a diffeomorphism since it is not a smooth manifold.

It is a singular.

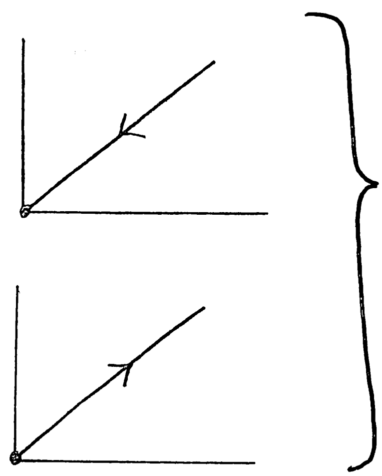
It is a homeomorphism.

Take another example.

$g : \mathbb{R}^1 \longrightarrow \mathbb{R}^2, \quad g(t) = (t^2, t^2)$

$g(\mathbb{R}^+) \Longrightarrow$

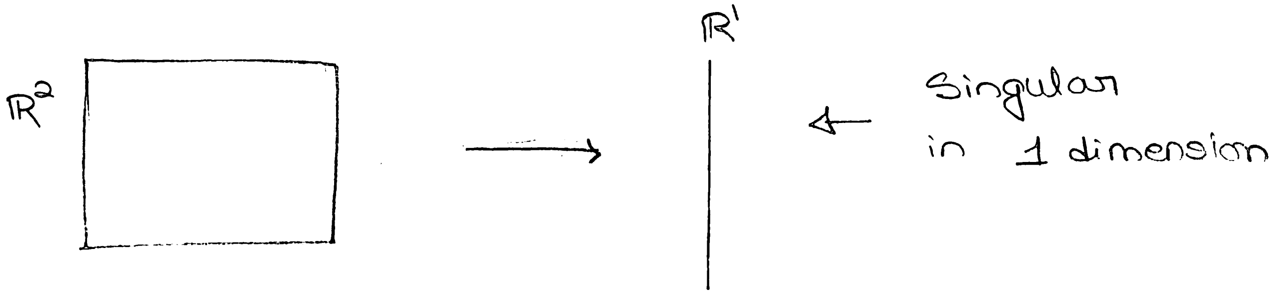
$g(\mathbb{R}^-) \Longrightarrow$

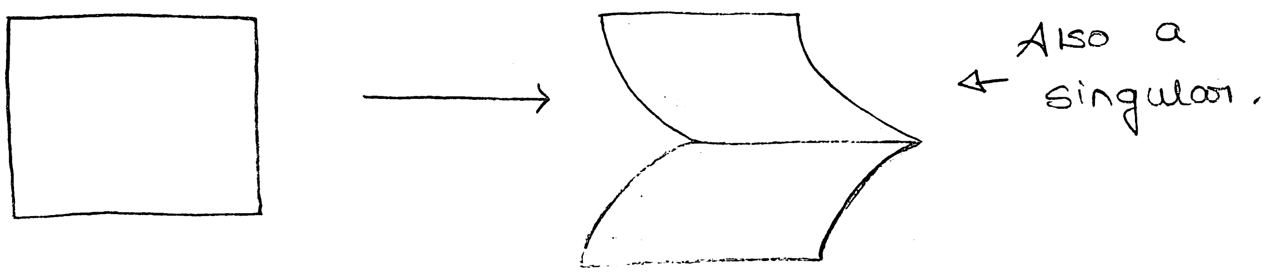


The graph traverses the same line twice.

This is therefore still no good.

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is a singular map $\mathbb{R}^2 \rightarrow \mathbb{R}^1$





Recall that Df is a map for an 'n' manifold.

$\therefore$ Df_x needs to have a rank 'n.'

i.e. For an n manifold in $\mathbb{R}^m$

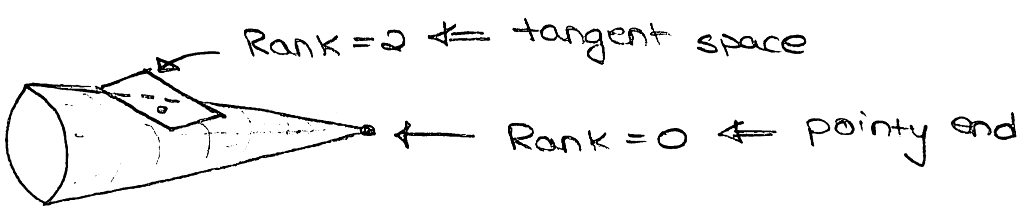
$$Df_x : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

Recall that rank n means there is an n-dimensional basis for a given matrix.

$\therefore$ Df_x has a rank n means there are n linearly independent columns and rows in the Df_x matrix.

If Rank $(Df_x) = 0$, $Df_x : \mathbb{R}^n \rightarrow \mathbb{R}^0 \Rightarrow$ point

Eg: (1)



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