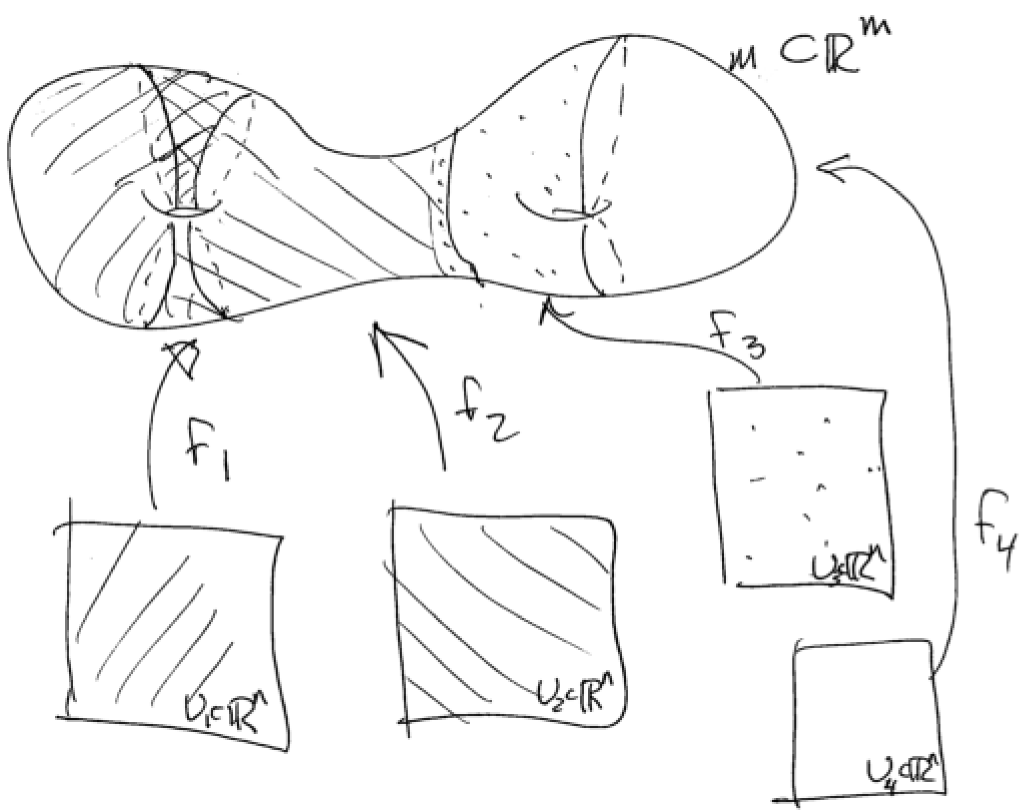
Notes 9/14
MAT 364

A function *f* is called *smooth* if is differentiable at every point and f^{-1} exists and is also differentiable at every point (Milnor p. 2).

A function *f* is a *diffeormorphism* if *f* is a homeomorphism which is also smooth (Milnor p.2).

$M \subset \mathbb{R}^m$ is a *smooth manifold* of dimension *n* if for each $x \in M$ there exists a neighborhood $W \subset M$ diffeomorphic to an open subset $U \subset \mathbb{R}^n$. Generally $m > n$.

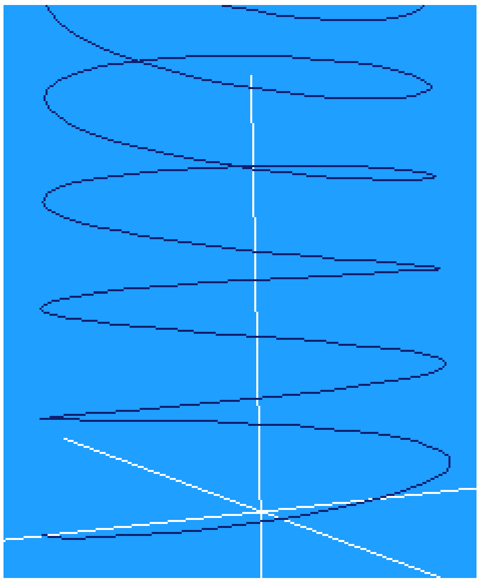
What this means is that if we zoom in close enough at any point $x \in M$, *M* will eventually look flat, i.e. linear. The dimension of this linear surface is the dimension of the manifold.

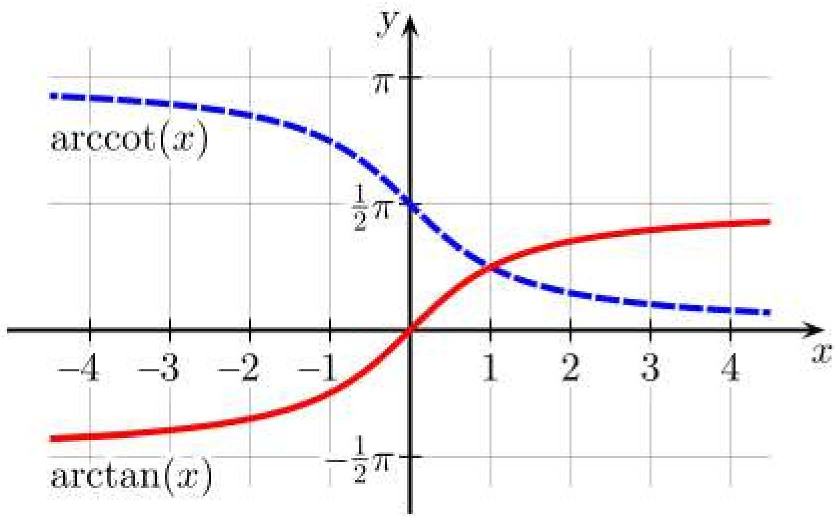


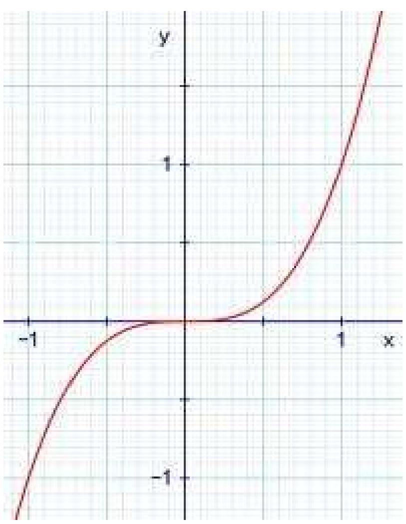
Each of these diffeomorphisms which send $U \subset \mathbb{R}^n$ to $W \subset M$ is called a *chart*. The union of these charts covers *M*. This idea of the union of charts covering the whole manifold can be seen in the same light as the video game example in class. Each screen of the video game can be seen as a different chart. By constructing the union of these screens, or charts, we can see the video game in its entirety.

By analyzing how each of these charts f_i changes, we can gain insight as to how our manifold M changes.

Here are some examples of homeomorphisms and diffeomorphisms.







In the first example above we have a curve in space. As was shown in class, we can break this curve up into sections and each of these sections will be diffeormorphic to some open interval. So, by definition, this is a manifold of dimension 1.

In the second example we have both the tangent and the cotangent, shown on the same set if axes. As discussed in class the function $f(x) = \tan(x)$ is a diffeomorphism from $\mathbb{R} \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2}) \subset \mathbb{R}$, and $g(x) = \cot(x)$ is a diffeomorphism from $\mathbb{R} \rightarrow (0,\pi)\ \mathbb{R}$.

The third example shows the function $f(x) = x^3$. This is a homeomorphism from $\mathbb{R} \rightarrow \mathbb{R}$., however it is not a diffiomorphism since at $x = 0$, the derivative of $f^{-1}(x) = \sqrt[3]{x}$ the derivative doesn't exist.

<u>The Derivative</u>

Recall the derivative if a function tells us the rate of change at each point in its domain. The definition is as follows:

$$\lim_{x \to a} f(x) - f(a) = D\,(x - a)$$

Where D is a linear map from $\mathbb{R}^n \rightarrow \mathbb{R}^m$. The definition must take this form since x, a ($\mathbb{R}^n$), $f(x)$, and $f(a)$ ($\mathbb{R}^m$) are vectors, and we cannot divide by a vector.

The derivative is denoted Df.

The derivative will be an m x n matrix with m being the dimension of the target space, or codomain, and n being the dimension of domain. Each entry of the matrix represents how each function changes with each variable.

The derivative is a linear mapping (perhaps the linearization of a non-linear function).

The image of $\mathbb{R}^n$ under the derivative map Df at x, is called the *tangent space* of M at x, and is denoted TM_x.

If M is a smooth manifold then at each point $x_0 \in M$ we have some chart:

$f : U \rightarrow x_0 \in M$

and we have the linearization Df which is a map from one vector space to another.

So, we bring $U \subseteq \mathbb{R}^n$ to a smooth manifold M in two ways:

1) Through $f(x)$ we bring each element of U to some element of M,
2) Through the linear mapping Df

Note that the derivative extends to all of $\mathbb{R}^n$, so that while f_i is a map from U into $\mathbb{R}^m$, Df is a map from $\mathbb{R}^n \to \mathbb{R}^m$ (this map is not necessarily onto since Df may be singular).