Topology Notes #1

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Abstract

Unless otherwise specified, we will be working in $\mathbb{R}^n$. The euclidean distance between x and y will be used, defined as $\sqrt{\sum (x_i - y_i)^2}$ and will be represented by $dist(x, y) = ||x - y||$.

What is Topology?

The concept of Topology grew out of geometry, so it's a good idea to understand what we're trying to study by understanding what geometry does. When we say two triangles are congruent, we mean that they have a "sameness" in the legnth of their sides and the measure of their angles, regardless of their orientation or location on the plane.

That property of "sameness" which Topology studies is what makes a line similar to a squiggle or a continuous function's graph in $\mathbb{R}^2$, or what makes a donut similar to a coffee cup or a human being. Specifically, one of the things we want to study are those homeomorphic topological spaces. There's a **homeomorphism** between two spaces A and B if we have a continuous, invertible function $f : A \to B$ with continuous inverse $f^{-1} : B \to A$.

In topology, we want to understand the idea of "closeness" or "connectedness" without using the concept of distance. However, we'll begin this using the idea of euclidean distance in $\mathbb{R}^n$.

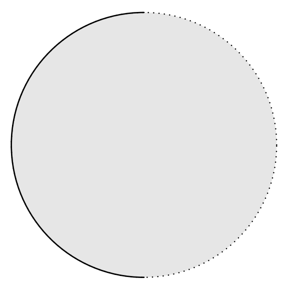
Beginning Definitions

Before we can start talking about homeomorphisms and all that nitty-grittiness of topology, we gotta get some jargon settled. An **Open Disk** of x is the set of all points less than a particular radius. Formally, the Open Disk $\mathrm{D}_r(x) = \{y \in \mathbb{R}^n | \ ||x - y|| < r\}$. For example, in $\mathbb{R}^1, \mathrm{D}_1(0)$ is the interval $(-1, 1)$. In $\mathbb{R}^2$, the open disk $\mathrm{D}_1((0, 0))$ is a filled-in unit circle without those points x, y such that $x^2 + y^2 = 1$. And if you can try to imagine a sphere in $\mathbb{R}^3$ without its very outer boundary, that's an open disk as well.

The following definitions are based upon the concept of the open disk. (Note that "iff" means "if and only if" and $\mathbb{R}^+$ are the strictly positive real numbers) We take a subset $A \subseteq \mathbb{R}^n$ and call a point $x \in A$

- an **Interior point** iff there's an $r \in \mathbb{R}^+$ so that $\mathrm{D}_r(x) \subset A$
- an **Exterior point** iff there's an $r \in \mathbb{R}^+$ so that $\mathrm{D}_r(x) \subset \mathbb{R}^n / A$
- a **Limit point** iff for all $r \in \mathbb{R}^+, \mathrm{D}_r \cap A \neq \emptyset$

Let's use the example of a "Half-open Disk" to understand each of these kinds of points. In $\mathbb{R}^2$, let $A = \mathrm{D}_1(0) \cup \{(x,y)|x^2+y^2=1 \text{ and } x \leq 0\}$



We can say that the point $(0,0)$ and $(.5,.5)$ are interior points because we can create an open disk that is a proper subset of A. What about the point $(2,0)$? It must be an exterior point, as we can create an open disk around it that's in the complement of A, that is, we can create a disk $B = \mathrm{D}_r((2,0))$ such that $B \subset \mathbb{R}^2$ but $B \not\subset A$.

We shall now define three sets based upon our set A.

- The **interior** of A is the set $\text{Int } A = \{x| \text{ x is an interior point of } A\}$
- The **exterior** of A is the set $\text{Ext } A = \{x| \text{ x is an exterior point of } A\}$
- The **limit** of A is the set $\text{Lim } A = \{x| \text{ x is a limit point of } A\}$

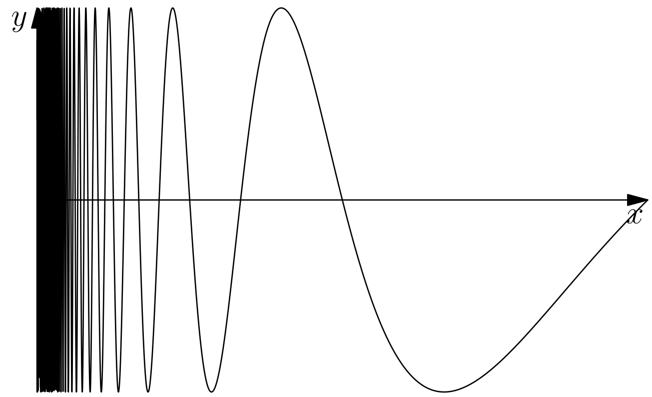
That settles all but one part of our half-open disk: What about the unit circle $S^1 = \{(x,y)|x^2+y^2=1\}$?

We can tell that all $p \in S^1$ are limit points of A, as any neighborhood on the outer circle of out half-disk must intersect with the gray area. But they aren't interior points, since all neighborhoods must also contain some exterior points. Therefore, $p \notin \text{Ext } A$ and $p \notin \text{Int } A$. Therefore, we need another definition:

- A **Frontier point** of A is a point x such that $x \in \text{Lim } A$ and $x \notin \text{Int } A$. The **frontier** is the set of points $\text{Fr } A = \{x| \text{ x is a frontier point of } A\}$.

So, according to that definition, the unit circle S^1 is all frontier.

To illustrate better what the frontier is, we will introduce the "Topologist's sine curve." In $\mathbb{R}^2$, let $G = \{(x,y)|x \in \mathbb{R}^+, y = \sin \dfrac{1}{x}\}$.



The set $L = \{(0, y) | -1 \leq y \leq 1\}$ is not a subset of G. However, $L = \operatorname{Fr} G$ because we can create a neighborhood around any point in L and hit a point in G. (Technically, infinitely many points in G) Note that $\operatorname{Lim} X = \operatorname{Int} X \cup \operatorname{Fr} X$ for all sets X.

Sidenotes

Here are two more definitions that really don't fit in anywhere but should be mentioned before we get into the concept of open sets. A point x is **isolated** iff there's an open disk so that $A \cap \mathrm{D}_r(x) = \{x\}$. For instance, in the $\mathbb{R}^1$ set $[0, 1] \cup \{2\}$, the point 2 is isolated.

A set $A \subset \mathbb{R}^n$ is **bounded** iff we can create an open disk around some point $x \in \mathbb{R}^n$ so that $A \subset \mathrm{D}_r(x)$. Basically, it means a set is not infinite in any dimension. Both of these concepts should be pretty intuitive.

Open Sets

- A set U is **Open** if $\operatorname{Int} U = U$.
- A set V is **Closed** if $\mathbb{R}^n / V$ is open.

To get an intuition for this, imagine a tiny town. The borders of this town are extremely well-defined, but there are no fences at all on the borders. This town is "open." Now imagine a big fence is put up around the town *exactly where the borders are*. The fence wraps around the town and has no gaps. Now, the town is "closed."

Therre are sets $K \neq \emptyset$ with $\operatorname{Int} K = \emptyset$. For example, the set of all integers $\mathbb{Z} \subset \mathbb{R}$ is a non-empty set whose interior is empty, i.e. for all $z \in \mathbb{Z}$ there is no $r > 0$ such that $\mathrm{D}_r(z) \subset \mathbb{Z}$. The set $\{x\}$ has no interior, either. However, the set $\emptyset$ is both open and closed; it has no interior, so it's equal to its interior (thus making it open), and $\mathbb{R}^n$ is open, so $\emptyset$ is closed.

At this point, I should mention that I explicitly tell what context we're working in for all the examples used. This is not trivial. The interval $(-1, 1)$

is open in $\mathbb{R}$, because any point $x \in (-1, 1)$ has a $\mathrm{D}_r \subset (-1, 1)$. However, in $mathbbR^2$, the set $I = \{(x, 0)| - 1 < x < 1\}$ has *no interior!* There are points outside I. Therefore, the context we're working in matters.

This is related to a concept called **Relatively Open Set** that will probably be in the next set of notes. Have you ever seen that Twilight Zone episode where this kid is psychic and lords over this town through fear? One of the features of this episode is that the kid has physically transported his entire town into this strange void. Essentially, the entire universe consists of this town. Topologically, we consider the town to be both open and closed relative to itself, regardless of any fence we put up. To make a more mathematically concrete example, the interval $[-1, 0) \subset \mathbb{R}^1$ is neither open nor closed, but if we take the interval relative to $[-1, 1)$, it's relatively open. This also means that $\mathbb{R}^n$ is both open and closed.

Final Properties

We define the **Closure** of A to be $\mathrm{Cl}\, A = A \cup \mathrm{Fr}\, A$. A proposition that's left for homework is to show that the Closure of any set must be closed.

Finally, a couple of facts about the union and intersection of open and closed sets. Let $\{A_i\}$ be a countably infinite set of open sets, and let $\{B_i\}$ be a countably infinite set of closed sets. We can prove:

$$\bigcup_{i=1}^{\infty} A_i \text{ is open,} \qquad \bigcap_{i=1}^{n} A_i \text{ is open}$$

$$\bigcup_{i=1}^{n} B_i \text{ is closed,} \qquad \bigcap_{i=1}^{\infty} B_i \text{ is closed}$$

However, we can make no guarentee that $\cap^{\infty} A_i$ will be open or $\cup^{\infty} B_i$ will be closed.