MAT360 Final Exam

May 19, 2009

Name: ______________________ **ID:** ________________

Question:	1	2	3	4	5	6	7	8	Total
Points:	15	10	10	10	10	10	10	0	75
Score:									

There are 8 problems in this exam. Make sure that you have them all.

Do all of your work in this exam booklet, and cross out any work that the grader should ignore. You may use the backs of pages, but indicate what is where if you expect someone to look at it. **This is an open-book, open-notes exam.** However, it is not an open-friends or open-internet exam: you may only refer to the textbook or papers that you brought with you. If you have a psychic connection with the spirit of Felix Klein or David Hilbert, you may use that if you wish.

Unless otherwise stated, all questions apply to a real Hilbert plane. That is, you may assume all thirteen of Hilbert's Axioms and Dedekind's axiom, but NOT a parallel axiom.

You have $2\frac{1}{2}$ hours to complete this exam. That should be plenty of time, but I know it won't be.

The study of non-Euclidean Geometry brings nothing to students but fatigue, vanity, arrogance, and imbecility. "Non-Euclidean" space is the false invention of demons, who gladly furnish the dark understanding of the "non-Euclideans" with false knowledge. The "non-Euclideans," like the ancient sophists, seem unaware that their understandings have become obscured by the promptings of the evil spirits.

—Matthew Ryan (1905)

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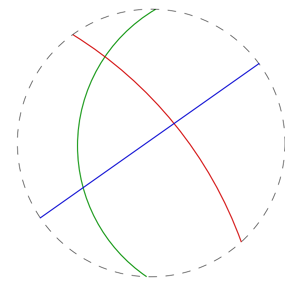
1. Consider the following interpretation of the terms “point”, “line”, and “incidence”:
   - A point is any pair of real numbers (x, y) such that $x^2 + y^2 < 1$. That is, any point of $\mathbb{R}^2$ lying inside the unit circle. The boundary of the circle is *not* included.
   - A line is either an open diameter of the unit circle or an (open) arc of a circle connecting two diametrically opposed points of the unit circle[1]. Note that such arcs run from a point (r, s) on the unit circle to another point $(-r, -s)$. Three such lines are shown at right.
   - Incidence is the usual relation of set membership.



$\boxed{\text{6 pt}}$ (a) Does this define an incidence geometry? That is, do axioms **I1**, **I2**, and **I3** hold? Fully justify your answer.

[1] If it helps you, such arcs are parts of circles satisfying the equation $(x-a)^2 + (y-b)^2 = a^2 + b^2 + 1 + \dfrac{2(a^2 - b^2)}{\sqrt{a^2 + b^2}}$, where a and b are any real numbers not both zero. But I doubt that will be useful.

$\boxed{\text{4 pt}}$ (b) Does this model define a Euclidian, elliptic, or hyperbolic geometry (or none of the those)? Again, fully justify your answer.

$\boxed{\text{5 pt}}$ (c) Suppose we change the model so that we *include* the unit circle in our allowable points. That is, points satisfy $x^2 + y^2 \leq 1$, and lines are now *closed* arcs or diameters (they include the endpoints).

How does this change your answer to parts (a) and (b) above? Be sure to answer and justify *both* parts.

You may skip **one** of problems 2 through 8 by crossing off the page.

$\boxed{\text{10 pt}}$ 2. Let $\square ABCD$ be a parallellogram, that is, lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel, as are $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$. Prove that **if the geometry is Euclidean**, then $\overline{AB} \cong \overline{CD}$. Note that none of the angles are required to be right angles.

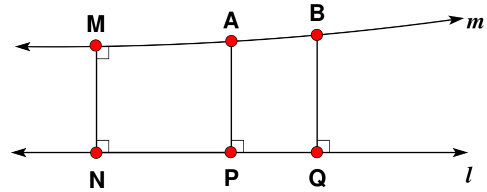
You may skip **one** of problems 2 through 8 by crossing off the page.

$\boxed{10\text{ pt}}$ 3. Prove that given triangle $\triangle ABC$ with $|\overline{AB}| > |\overline{BC}|$, we have $\angle C > \angle A$. Obviously, you can't use Proposition 4.5 ("the greater angle is opposite the greater side") in your proof.
Hint: find D on $\overline{AB}$ so that $\overline{DB} \cong \overline{CB}$.

You may skip **one** of problems 2 through 8 by crossing off the page.

10 pt 4. In a real hyperbolic plane, let lines l and m be divergently parallel with common perpendicular $\overleftrightarrow{MN}$.

Let A and B be any two points on m so that $M \star A \star B$, and let P and Q be points on l so that $\overline{AP} \perp l$ and $\overline{BQ} \perp l$.
Prove that $|\overline{AP}| < |\overline{BQ}|$.



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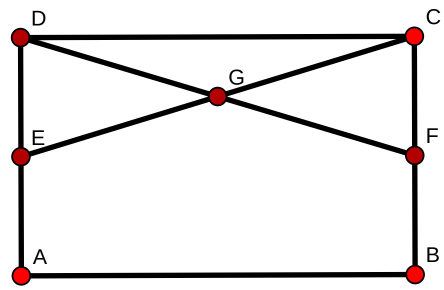
10 pt 5. Let $\square ABCD$ be a Saccheri quadrilateral, with right angles at A and B, and $\overline{AD} \cong \overline{BC}$. Also, let E and F be the midpoints of $\overline{AD}$ and $\overline{BC}$ respectively, and let G be the point of intersection of $\overline{EC}$ and $\overline{DF}$. Prove that if G is the midpoint of $\overline{EC}$ and $\overline{FD}$, then the geometry is Euclidean.



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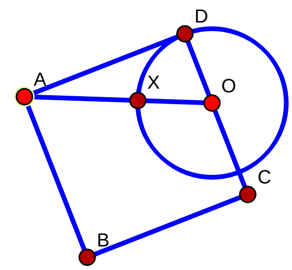
$\boxed{\text{10 pt}}$ 6. In Euclidean geometry, consider a square $\square ABCD$ where $\overline{AD}$ is tangent to a circle γ center O and radius $\overline{OD}$. Let X be the point where $\overline{AO}$ crosses γ.
Show that the area of $\square ABCD$ is equal to

$$|\overline{AX}|^2 + 2|\overline{AX}| \cdot |\overline{OX}|$$



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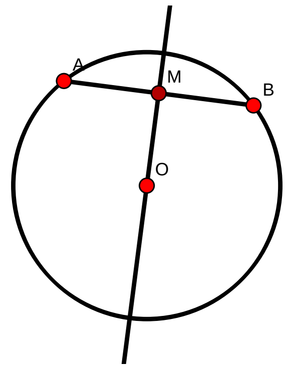
$\boxed{\text{10 pt}}$ 7. Let γ be a circle with center O, and let A and B be two points on γ.



Let M be the midpoint of $\overline{AB}$.
Prove that if $O \neq M$, then $\overleftrightarrow{OM}$ is perpendicular to $\overleftrightarrow{AB}$.

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10 pt 8. Justify each step in the proof of the ASA congruence theorem. We are given $\triangle ABC$ and $\triangle DEF$, with $\angle A \cong \angle D$, $\overline{AC} \cong \overline{DF}$, and $\angle C \cong \angle F$.

1. There is a unique point G on ray $\overrightarrow{DE}$ such that $\overline{DG} \cong \overline{AB}$.

2. $\triangle ABC \cong \triangle DGF$.

3. $\angle DFG \cong \angle C$.

4. Thus, $\overrightarrow{FE}$ and $\overrightarrow{FG}$ are the same ray.

5. Hence $G = E$.

6. $\triangle ABC \cong \triangle DEF$.