MAT360 Solutions to Homework

1. (Chapter 4,#10)

   Prove Proposition 4.7: Hilbert's Euclidean parallel postulate $\Leftrightarrow$ If a line intersects one of two parallel lines, it must also intersect the other.

   Deduce a corollary that transitivity of parallelism is equivalent to Hilbert's Euclidean parallel postulate.

   Solution: First we show that Hilbert's parallel postulate implies the line crossing condition.

   Suppose we have two parallel lines l and m, and another line t which intersects l. Let P be the point where l and t intersect. We must show there is a point Q where m and t intersect. But if there is no such Q, then lines t and m are parallel. However, by Hilbert's parallel postulate, there is at most one line which is parallel to m and contains the given point P. Thus we have a contradiction.

   Now let us show the converse: for any pair of parallel lines l and m, we know that if a line t crosses one, it must also cross the other. We must establish that Hilbert's parallel postulate holds under this assumption. So let l be a given line, and P be a point not on it; we must show there are not two lines m and n passing through P which are parallel to l. If so, then $m \parallel l$, and so since n crosses m at P, it must cross l. But this contradicts the assumption that l and n were parallel.

   Now we are to deduce that transitivity of parallelism is equivalent to Hilbert's parallel postulate. That is,

   $$(l \parallel m \text{ and } m \parallel n \Rightarrow l \parallel n) \Leftrightarrow \text{Hilbert's parallel axiom.}$$

   But observe that the contrapositive of transitivity of parallelism is precisely the statement we dealt with before. That is, the contrapositive is

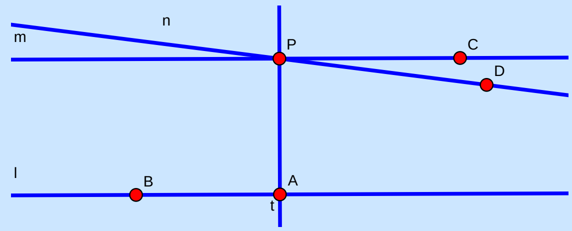
   $$l \nparallel n \Rightarrow l \nparallel m \text{ or } m \nparallel n$$

   or, in words, "if l crosses n, then either l crosses m or n crosses m". If we use t instead of n and assuming $l \parallel m$, we have the statement above: "if l crosses t, then t crosses m" (since we cannot have l crossing m).

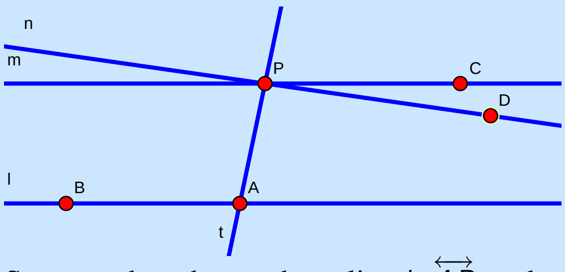
2. (Ch. 4, #11) Prove that Hilbert's parallel postulate is equivalent to the converse of the Alternate Interior Angles theorem.

   Solution: First, we assume the converse to AIA, and establish Hilbert's parallel postulate. We have a line l and a point P, and want to demonstrate that there is at most one line parallel to l containing P.

Construct a perpendicular t to l that contains P, and then let m be the line perpendicular to t. For notational purposes, let A be the point where t and l intersect, B be another point l, and C be a point of m on the other side of t from B. (See the figure).



Now suppose that Hilbert's parallel postulate fails to hold, that is, there is another line n which contains P and is parallel to l. Let D be a point on n on the opposite side of $\overleftrightarrow{AP}$ from B. By the converse of AIA, since $n \parallel l$ and $\angle BAP$ and $\angle APD$ are alternate interior angles, $\angle BAP \cong \angle APD$. But this means that $m = n$, by congruence axiom C4.



For the other direction, we assume Hilbert's parallel postulate and show that whenever two parallel lines l and m are cut by a transversal t, the resulting alternate interior angles are congruent.

Suppose then that we have line $l = \overleftrightarrow{AB}$ cut by a transversal $t = \overleftrightarrow{AP}$, with $n = \overleftrightarrow{PD}$ being parallel to $\overleftrightarrow{AB}$. Suppose also, for contradiction, that the alternate interior angles $\angle BAP$ and $\angle APD$ are not congruent. Then, by axiom C4, we can create line $\overleftrightarrow{PC}$ so that $\angle BAP \cong \angle APC$. Applying the Alternate Interior Angle theorem (not the converse!), we know that $\overleftrightarrow{AB} \parallel \overleftrightarrow{PC}$. But this contradicts Hilbert's postulate, since we have two lines containing P that are parallel to l.

3. (Ch 4. #14) Fill in the details of Heron's proof of the triangle inequality (for any triangle $\triangle ABC$, we have $|\overline{AB}| + |\overline{AC}| > |\overline{BC}|$).

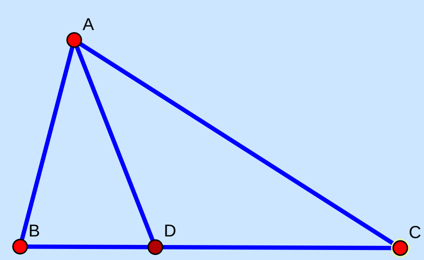
Solution: Given $\triangle ABC$, bisect $\angle A$, and let the bisector meet $\overline{BC}$ at a point D (which must exist because of the crossbar theorem).

Observe that $\angle ADC$ is an exterior angle to $\triangle ABD$, so $\angle ADC > \angle BAD$, but $\angle BAD = \angle DAC$ (since we bisected the angle at A). Thus, $|\overline{AC}| > |\overline{DC}|$ since in any triangle, the greater angle is opposite the longer side.

Similarly, $\angle ADB$ is an exterior angle to $\triangle ADC$, so $\angle ADB > \angle DAC = \angle BAD$, and thus $|\overline{AB}| > |\overline{BD}|$. Adding these together gives

$$|\overline{AB}| + |\overline{AC}| > |\overline{BD}| + |\overline{DC}| = |\overline{BC}|,$$

as desired.

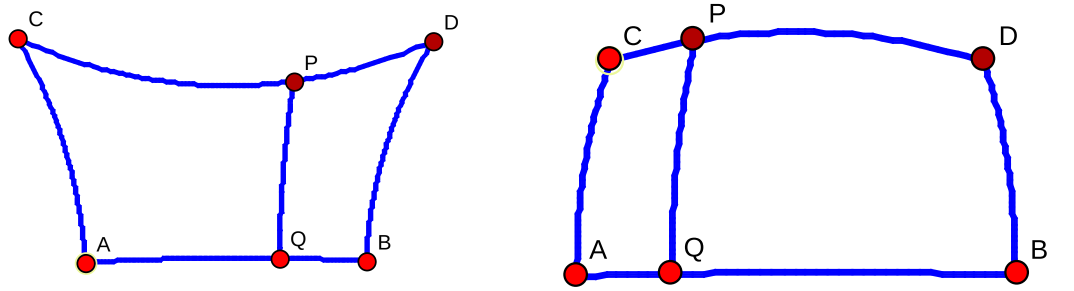


4. (Ch.4, Major Ex. 5) Given a Saccheri quadrilateral $\square ABCD$ and a point P between C and D. Let Q be the foot of the perpendicular from P to the base $\overline{AB}$. Then show that

   (a) $|\overline{PQ}| < |\overline{BD}|$ if and only if the summit angles of $\square ABCD$ are acute.

   (r) $|\overline{PQ}| = |\overline{BD}|$ if and only if the summit angles of $\square ABCD$ are right.

   (o) $|\overline{PQ}| > |\overline{BD}|$ if and only if the summit angles of $\square ABCD$ are obtuse.



Solution: Since $\square ABCD$ is a Saccheri quadrilateral, we know that the summit angles $\angle C$ and $\angle D$ are congruent, and that $\overline{AC} \cong \overline{BD}$. Also observe that $\square AQPC$ and $\square QBDP$ are bi-right quadrilaterals; thus we can apply the "greater angle is opposite the greater side" theorem (Prop. 4.13).

Note that angles $\angle CPQ$ and $\angle DPQ$ are supplementary; thus they are either both right angles, or one is acute and one is obtuse. Without loss of generality, we may assume that $\angle CPQ \leq \angle DPQ$. If they are equal, both are right angles, and if not, then we will assume $\angle CPQ$ is acute and $\angle DPQ$ is obtuse.

First, let us establish the forward direction of all three cases:

Suppose $|\overline{PQ}| < |\overline{BD}|$. Since $|\overline{BD}| = |\overline{AC}|$ by hypothesis, we can apply the "greater angle/longer side" theorem to see that $\angle C < \angle CPQ$. But $\angle CPQ \leq 90°$, and so $\angle C$ must be acute. Hence the summit angles are acute.

Now if $|\overline{PQ}| > |\overline{BD}|$, then we know that $\angle D > \angle DPQ$. But since $\angle DPQ \geq 90°$, $\angle D$ is obtuse.

Finally, if $|\overline{PQ}| = |\overline{BD}|$, then $\angle D$ and $\angle QPD$ are congruent, as are $\angle C$ and $\angle QPC$. But since $\angle D \cong \angle C$, we know $\angle QPC \cong \angle QPD$. Since these are supplementary angles, they must be right.

Now we establish the reverse direction, which works in much the same way.

If $\angle D$ is acute, then since $\angle QPD$ is not acute, we have $\angle D < \angle QPD$, and so $|\overline{PQ}| < |\overline{BD}|$.

If $\angle C$ is obtuse, then since $\angle QPC$ is not obtuse, $\angle C > \angle QPC$, and so $|\overline{PQ}| > |\overline{AC}| = |\overline{BD}|$.

If the summit angles are right, then $\square AQPC$ and $\square QBDP$ are Lambert quadrilaterals, and so by Cor. 3 to Prop. 4.13, $\overline{PQ} \cong \overline{BD}$. (In fact, these must all be rectangles).