MAT331 homework problems

17. (*expires 18 March*) Consider the differential equation $\dot{\mathbf{z}}(t) = \mathbf{F}(\mathbf{z}(t))$, where the vector $\mathbf{z}(t) = (x(t), y(t))$ and the field $\mathbf{F}(x,y) = (-y, x-y)$. Plot a few solutions. What happens to them when $t \to +\infty$? Give a "`Maple`-proof" that this is a general fact for *every* solution. [*A "`Maple`-proof" is an argument that is rigorous once we accept `Maple` results as incontrovertibly true.*]

18. (*expires 18 March*) (No `Maple`.) For the equation $\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z})$, $\mathbf{z} = (x,y)$, with the vector field

$$\mathbf{F}(x,y) = \left\langle -x(x^4+y^4) - y \, , \, x - y(x^4+y^4) \right\rangle ,$$

prove that the origin is an attractor in the future, i.e., every solution verifies

$$\lim_{t \to +\infty} \mathbf{z}(t) = 0.$$

[*You can ask around how to do this, but then you have to show clearly that you have understood it.*]

19. (*expires 18 March*) We will study the Lotke-Volterra predator-prey equations: In a very simple ecosystem, at the time t (which is expressed, say, in years), there is a population of $f(t)$ foxes and $r(t)$ rabbits. The evolution of these quantities obeys the system

$$\begin{cases} \dot{f}(t) &=& G_f\, f(t) + E\, f(t)\, r(t), \\ \dot{r}(t) &=& G_r\, r(t) - E\, f(t)\, r(t); \end{cases}$$

where G_f and G_r are the growth rates for the foxes and the rabbits, respectively, in the absence of each other. E is the probability of a fatal encounter between a fox and a rabbit (normalized per number of foxes and rabbits).

First, write some words to explain why these equations make sense. Then, fix $G_f = 0.4$, $G_r = 2.4$ (it's notorius that rabbits have the tendency to reproduce quickly) and $E = 0.01$. For a few initial conditions of your choice, plot the trajectories in the (f,r)-plane (say, with $0 \le f \le 1000$ and $0 \le r \le 1000$). For the same initial conditions, plot the actual solutions too (i.e, $f(t)$ against t, and $r(t)$ against t). Write some comments interpreting how the behaviour of the solutions relates to what happens to the two species.

Finally, repeat the same procedure with $G_f = -1.1$. Things change substantially. Again, what is the "physical" interpretation of this?