MAT 331 Homework Exercises, 18-22

NOTE: *Each exercise is worth 10 points and can be turned in at any time before its "expiration date". You can work on any number of problems per homework sheet (none to all, usually 2 to 3). However, at the end of the semester, we will expect you to have turned in at least 2/5 of the exercises assigned. If you do more, we will pick your best grades. If you do less, the missing grades will be counted as zeros. This will determine 20% of your final grade for the class.*

If you don't like Maple as a programming tool, you can use C/C++, Java or Perl for any problem in this assignment. Submit the source code which can be compiled and executed. Comment your code.

#18 (exp. 12/01) Newton's method is an algorithm for finding roots of polynomial equations. The basic idea behind the method is the following:

1. Start with a function f and an initial value x_0.
2. Check whether $f(x_0) = 0$. If it does, the process stops. If not go to step 3.
3. Check whether $f'(x_0) = 0$. If it does, Newton's method fails. If not, set $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$. That is, x_1 is the x-intercept of the tangent line to $y = f(x)$ at the point $(x0, f(x0))$.
4. Repeat from step 2 on with x_1 replacing x_0.
5. In general after the i-th-iteration of Newton's method we check whether $f(x_i) = 0$, if so we've found a root. If not we check whether $f'(x_i) = 0$, then we set

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Note that $x_{i+1} = x_i$ if and only if $f(x_i) = 0$ (and $f'(x_i)$ is not equal to zero). Therefore, we can also set up the algorithm to just check whether $x_{i+1} = x_i$. Also due to rounding error, it makes sense to compare only *closeness* of $f(x_i)$ to zero, but not exact equality.

Using Maple write a program that will find real roots of the equation

$$f(x) = \frac{7}{8}x^5 + \frac{23}{2}x^4 + \frac{459}{8}x^3 + \frac{541}{4}x^2 + 151x + 65 \tag{1}$$

Make a judicial choice of initial points. Do not use built-in Maple functions related to solving equations.

#19 (exp. 12/01) Explore the situation when the choice of initial value is $x_0 = -3$. Use do loop to find the trajectory of this point under Newton's iterations. Explain what happens.

#20 (exp. 12/01) This problem demonstrates the chaotic nature of Newton's method. Explore the situation when we perturb the initial point: let $x_0 = -3 + 10^{-9}$ and $x = -3 - 10^{-9}$. Speculate about the *stability* of the Newton's method, that is how well the methods depend on the initial value.

#21 (exp. 12/01) One can try to generalize Newton's method for solving polynomial equations of one complex variable (using the same algorithm). Do that for polynomial $f(z) = z^2 + 1$. Can you guess which initial points will converge to which root?

#22 (exp. 12/01) For polynomials of higher order, the attractor for each root (i.e. the set of points that converge to the root under Newtons iterations) is often of fractal nature, similar to the Julia set of a quadratic polynomial. Modify the program that was used in class for drawing Mandelbrot set for finding the attractor of $z = 1$ for the polynomial $p(z) = z^3 - 1$.