# MAT319/320 Solutions to First Midterm

1. [10 points] Give a careful and complete definition of what it means when we say "The limit of the sequence $X$ is $L$."

**Solution:** The limit of the sequence $X = (x_n)$ is $L$ if for any $\epsilon > 0$, there is a natural number $K_\epsilon$ so that $|x_n - L| < \epsilon$ for all $n \geq K_\epsilon$.

2. [10 points] Let $A$ and $B$ be bounded subsets of $\mathbb{R}$.

(a) Prove that $A \cup B$ is a bounded subset of $\mathbb{R}$.

**Solution:** Since $A$ is bounded, there are real numbers $u_A$ and $l_A$ so that every element of $A$ lies between $u_A$ and $l_A$; that is, $A \subseteq [l_A, u_A]$. Similarly, there are real numbers $l_B$ and $u_B$ so that $B \subseteq [l_B, u_B]$.

Let $L = \inf(l_A, l_B)$, and let $U = \sup(u_A, u_B)$. Then certainly $A \subseteq [L, U]$, and also $B \subseteq [L, U]$. If $x \in A \cup B$, then $x \in A$ or $x \in B$, and so $x \in [L, U]$. Thus $A \cup B \subseteq [L, U]$, which means $A \cup B$ is bounded.

(b) Prove that $\sup(A \cup B) = \sup(\sup A, \sup B)$.

**Solution:** Let $U = \sup(\sup A, \sup B)$. We must show that for any $x \in A \cup B$, we have $x \leq U$, and we must also show that if $V < U$, then there is a $y \in A \cup B$ so that $y > V$.

The first part is nearly immediate: since $U \geq \sup A$, for every $a \in A$ we have $a \leq U$. Similarly, since $U \geq \sup B$, we have $b \leq U$ for every $b \in B$.

For the second part, note that either $U = \sup A$ or $U = \sup B$. Without loss of generality, suppose the former holds. Then since $U$ is the supremum of $A$, if $V < U$, there is an $a \in A$ so that $a > V$. Since this same $a$ is an element of $A \cup B$, we have the desired conclusion.

3. (a) [10 points] Prove that for all natural numbers $n$, we have $2^n \geq n + 1$. You might find induction helpful.

**Solution:** First, we see that for $n = 1$, we have $2 = 2^1 \geq 1 + 1$, so the base case holds.

Now for the inductive step, we want to show that $2^k \geq k + 1$ implies that $2^{k+1} \geq k + 2$. We have

$$\begin{aligned} 2^{k+1} = 2 \cdot 2^k &\geq 2(k+1) \qquad \text{by our inductive hypothesis} \\ &= 2k + 2 > k + 2, \end{aligned}$$

where we have used $k > 0$ in the final step.

This shows that $2^n \geq n + 1$ for all natural numbers $n$.

(b) Prove that for all natural numbers $n \geq 4$, we have $2^n \geq n^2$.
Feel free to use the result from part a, even if you couldn't do it.

**Solution:** First, we establish the relation for $n = 4$, our base case. We have $2^4 = 16 = 4^2$, as desired.

Now we show that if $2^k \geq k^2$, then we also have $2^{k+1} \geq (k + 1)^2$.

Unfortunately, it seems we need to establish that $2^k \geq 2k + 1$ first, so let's do that, again by induction. First, we see that $2^4 = 16 > 9 = 2 \cdot 4 + 1$. For the inductive step, we must establish that $2^{k+1} \geq 2(k + 1) + 1$. So

$$\begin{aligned} 2^{k+1} = 2 \cdot 2^k \geq & 2 \cdot (2k + 1) \qquad \text{using the induction hypothesis} \\ = & 4k + 2 = 2k + 2k + 2. \end{aligned}$$

Since $k > 1$, we have $2k > 2$ and so we have

$$2k + 2k + 2 > 2k + 4 > 2(k + 1) + 1.$$

This is our desired result.

Now we return to our main proof:

$$2^{k+1} = 2^k + 2^k \geq k^2 + (2k + 1) = (k + 1)^2.$$

Here we used the induction hypothesis ($2^k \geq k^2$) and our previous fact ($2^k \geq 2k + 1$).

I'm sorry that a typo in part a caused it not to be useful (although still true).

4. 10 points Consider the sequence whose $n^{th}$ term $a_n = \left(\frac{-1}{2}\right)^n$. Prove, using the definition of the limit, that the limit of this sequence is $0$.

**Solution:** Given $\epsilon > 0$, we need to find a $K$ so that $|a_n| < \epsilon$ for all $n \geq K$.

Note that from the previous problem, we know that $2^n \geq n + 1 > n$, and so if we take $K \geq 1/\epsilon$, for any $n > K$ we will have

$$\left|\left(\frac{-1}{2}\right)^n\right| = \frac{1}{2^n} \leq \frac{1}{n} \leq \frac{1}{1/\epsilon} = \epsilon,$$

as desired.

5. 10 points Let $f : (0, 1) \to \mathbb{R}$ have the property that $f(x) < x$ for all $x \in (0, 1)$.

(a) Prove that $\sup_{x \in (0,1)} f(x) \leq 1$

**Solution:** Suppose not. If the supremum is greater than $1$, there must be some some $x \in (0, 1)$ with $1 < f(x)$. But $f(x) < x < 1$, and so this is a contradiction.

(b) Is it true that $\sup_{x \in (0,1)} f(x) < 1$? Prove or give a counterexample.

**Solution:** This is not true.

There are plenty of counterexamples, such as $f(x) = x^2$ or $f(x) = 2x - 1$.

Let's use the second one. For $f(x) = 2x - 1$, we certainly have $f(x) < x$ for all $x < 1$. But the image of the interval $(0, 1)$ under $f$ is $(-1, 1)$, and so $\sup f(x) = 1$.