Correction of Practice Midterm I

Problem 1. Show by induction that for any natural number $n \geqslant 1$ one has $3^n > n$.

Proof.

1. The proposition is true for $n = 1$ because $3 > 1$.
2. Assume that $3^n > n$: then $3^{n+1} > 3n = n + 2n > n + 1$ because $n \geqslant 1$.Therefore we proved that $3^{n+1} > n + 1$.

□

Problem 2. Determine the set A of all x in $\mathbb{R}$ such that $|5.x + 2| < 8$.

Proof. $|5.x + 2| < 8$ is equivalent to $-8 < 5x + 2 < 8$ whic is equivalent to $-10 < 5x < 6$, therefore the set A is the open interval $(-2, 6/5)$.

□

Problem 3. Is the set $B = \left\{\frac{1}{n^2+1}, n \in \mathbb{N}\right\}$ bounded above? bounded below? Does it have a least upper bound, a greatest lower bound?

Proof. For any $n \in \mathbb{N}$ one has $0 \leqslant \frac{1}{n^2+1} \leqslant 1$ therefore it is bounded above and below. Since it is nonempty it has a least upper bound and a greatest lower bound (by completeness of $\mathbb{R}$).Since 1 is an upper bound and is in the set B it is also its least upper bound.

Let's prove now that 0 is the infimum of B.Given any $\varepsilon > 0$, using the archimedean property one can find an integer K such that $\frac{1}{K} < \varepsilon$, but then for any $n \geqslant K$, one has:

$0 < \frac{1}{n^2+1} < \frac{1}{n^2} < \frac{1}{n} \leqslant \varepsilon$ therefore $\varepsilon = 0 + \varepsilon$ is not a lower bound of B, hence 0 is the greatest lower bound of B.

□

Problem 4. Let $J_n = (1 - \frac{1}{n}, 2 + \frac{1}{n})$. Prove that $\bigcap_{n=1}^{\infty} J_n = [1, 2]$.

Proof. Since $[1, 2] \subset J_n$ for any n, one deduces that $[1, 2] \subset \bigcap_{n=1}^{\infty} J_n$.

Let's prove the other direction: let $x \in \bigcap_{n=1}^{\infty} J_n$. We will successively prove that $x < 1$ and $x > 2$ are impossible:

Assume that $x < 1$. Then by the archimedean property, there is an integer K such that $\frac{1}{K} < 1 - x$, but this implies that $x < 1 - \frac{1}{K}$, therefore $x \notin J_K$, and thus $x \notin \bigcap_{n=1}^{\infty} J_n$.

Similarly assume that $x > 2$, then one can find an integer K such that $x > 2 + \frac{1}{K}$ and thus $x \notin J_K$ and we are done.

□

Problem 5. Prove that $\left(\lim\left(\frac{5n+3}{n+7}\right) = 5\right)$.

Proof. We have $0 \leqslant \left|\frac{5n+3}{n+7} - 5\right| = \left|\frac{5n+3-5n-35}{n+7}\right| = \left|\frac{-32}{n+7}\right| \leqslant \frac{32}{n}$.

Thus we can apply our criterion of convergence, because we already know that the sequence $\left(\frac{32}{n}\right)$ converges to zero. □

Problem 6. Find the limit of $\sqrt{(1 + \frac{1}{n^2+5}).(\frac{2n+1}{n^2+7} + 2)}$.

Proof. Let's break the proof into several steps:

1. $\left(\mathbf{1} + \frac{\mathbf{1}}{\mathbf{n^2+5}}\right)$ **converges to 1**:

Indeed, since we have $0 \leqslant \frac{1}{n^2+5} \leqslant \frac{1}{n} \to 0$, we know that $\left(\frac{1}{n^2+5}\right)$ converges to zero which gives the result.

2. $\boldsymbol{\left(\frac{2n+1}{n^2+7}+2\right)}$ **converges to 2:**

   Because for any $n \geqslant 1$ we have $0 \leqslant \frac{2n+1}{n^2+7} = \frac{2n}{n^2+7} + \frac{1}{n^2+7} \leqslant \frac{2n}{n^2} + \frac{1}{n^2} \leqslant \frac{2}{n} + \frac{1}{n} = \frac{3}{n} \to 0$

   one deduces by our convergence criterion that $\left(\frac{2n+1}{n^2+7}\right)$ converges to zero, and this gives the result.

3. By the product theorem, one deduces that $(1+\frac{1}{n^2+5}).(\frac{2n+1}{n^2+7}+2)$ converges to $1.2=2$.

   Since for any $n \in \mathbb{N}$ one has that $(1+\frac{1}{n^2+5}).(\frac{2n+1}{n^2+7}+2) \geqslant 0$ and converges to 2, one knows that $\sqrt{(1+\frac{1}{n^2+5}).(\frac{2n+1}{n^2+7}+2)}$ converges to $\sqrt{2}$.

$\square$