Solutions to Second Midterm Exam $\qquad$ MAT 310, Spring 2005

Name: $\qquad$ **ID#:** $\qquad$ **Rec:**

problem	1	2	3	4	5	Total
possible	25	25	25	25	25	100
score						

Directions: There are 5 problems on six pages (including this one) in this exam. Make sure that you have them all. Do all of your work in this exam booklet, and cross out any work that the grader should ignore. You may use the backs of pages, but indicate what is where if you expect someone to look at it.

Do any four problems. Cross out the one you don't want graded.

You may use any bound books or a calculator to do this exam. Using extra papers, notes, computers, or discussions with friends (or enemies) is not permitted. If you wish to make use of a time machine to look at the solutions, you may do so provided you admit such usage below, and then allow me to use it to retroactively change the questions.

Failure to admit such usage of a time machine will be grounds for charges of academic dishonesty (as is more "ordinary" methods of cheating).

1. *(25 points)* Let $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ satisfy $T\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}3\\7\end{smallmatrix}\right)$ and $T\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)$.

a) Write a matrix which represents T in the standard basis.

Solution: We need to establish the values of $T\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $T\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, and then the matrix will be immediate. Notice that

$$\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}-1\\1\end{pmatrix}\right]$$

and so, using the linearity of T, we have

$$T\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\left[T\begin{pmatrix}1\\1\end{pmatrix} - T\begin{pmatrix}-1\\1\end{pmatrix}\right] = \frac{1}{2}\left[\begin{pmatrix}3\\7\end{pmatrix} - \begin{pmatrix}-1\\1\end{pmatrix}\right] = \begin{pmatrix}2\\3\end{pmatrix}.$$

Similarly, since $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = \frac{1}{2}\left[\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) + \left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)\right]$ we have

$$T\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\left[T\begin{pmatrix}1\\1\end{pmatrix} + T\begin{pmatrix}-1\\1\end{pmatrix}\right] = \frac{1}{2}\left[\begin{pmatrix}3\\7\end{pmatrix} + \begin{pmatrix}-1\\1\end{pmatrix}\right] = \begin{pmatrix}1\\4\end{pmatrix}.$$

Therefore, the matrix of T in the standard basis is

$$\begin{pmatrix}2 & 1\\3 & 4\end{pmatrix}$$

b) Write a matrix which represents T in the basis $\left\{\binom{1}{1}, \binom{-1}{1}\right\}$.

Solution: Recall that if $T(\alpha) = a\alpha + c\beta$ and $T(\beta) = b\alpha + d\beta$, then the matrix representation of T the basis $\{\alpha, \beta\}$ is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

So, we just need to express $\binom{3}{7}$ and $\binom{-1}{1}$ as combinations of the vectors $\left\{\binom{1}{1}, \binom{-1}{1}\right\}$.

We have

$$\begin{pmatrix} 3 \\ 7 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

so, in the basis $\left\{\binom{1}{1}, \binom{-1}{1}\right\}$, T has the matrix form

$$\begin{pmatrix} 5 & 0 \\ 2 & 1 \end{pmatrix}.$$

2.*(25 points)* Let T be the transformation whose matrix in the standard basis is $\begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

a) What is the characteristic polynomial of T?

Solution: The characteristic polynomial is

$$\begin{aligned} \det \begin{pmatrix} x-2 & -1 & -3 \\ -1 & x-2 & -3 \\ 0 & 0 & x-1 \end{pmatrix} &= (x-1) \det \begin{pmatrix} x-2 & -1 \\ -1 & x-2 \end{pmatrix} \\ &= (x-1)(x^2 - 4x + 3) \\ &= (x-1)(x-1)(x-3) \\ &= (x-1)^2\,(x-3) \end{aligned}$$

b) What is the minimal polynomial of T?

Solution: The minimal polynomial must either be $(x-1)(x-3)$ or $(x-1)^2(x-3)$, so we just check whether $(T-I)(T-3I)$ is the zero matrix or not.

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 3 \\ 1 & -1 & 3 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence, the minimal polynomial is $(x-1)(x-3)$.

c) Is T similar over $\mathbb{R}$ to a diagonal matrix A? While you must justify your answer, you don't have to exhibit A if your answer is yes. An answer of "yes" or "no" without justification will receive at most one point of the nine possible.

Solution: Since the minimal polynomial is a product of unique linear factors, T is diagonalizable over $\mathbb{R}$.

3. *(25 points)* Let $\mathcal{P}$ be the infinite dimensional vector space of all polynomials with real coefficients, and let $f \in \mathcal{L}(\mathcal{P}, \mathbb{R})$ be the linear functional given by

$$f(p) = \int_0^1 p(x)\, dx$$

If $D \in \mathcal{L}(\mathcal{P}, \mathcal{P})$ is the differentiation operator (ie, $D(p) = p'$), what is $D^t(f)$?
Here D^t denotes the transpose of D.

Solution: For any polynomial $p \in \mathcal{P}$, we have

$$D^t(f)(p) = f(D(p)) = \int_0^1 p'(x)\, dx = p(1) - p(0).$$

4. *(25 points)* Let A be a 3×3 matrix with real entries. Prove that if A is **not** similar over $\mathbb{R}$ to an upper-triangular matrix, then A must be diagonalizable over $\mathbb{C}$.

Solution: If A is not similar over $\mathbb{R}$ to an upper-triangular matrix, its characteristic polynomial does not factor completely over $\mathbb{R}$. Since A is a 3×3 matrix, the characteristic polynomial $p_A(x)$ is of degree 3. Every cubic polynomial must have at least one real root; since $p_A(x)$ doesn't factor, it has exactly one. Therefore,

$$p_A(x) = (x-a)(x^2 + bx + c) \qquad \text{where } b^2 - 4c < 0.$$

Over the complex numbers, $p_A(x)$ has roots

$$x = a,\ x = \frac{-b + \sqrt{b^2 - 4c}}{2},\ x = \frac{-b - \sqrt{b^2 - 4c}}{2}$$

which are all distinct (since $b^2 \neq 4c$). Therefore, since A has distinct eigenvalues over $\mathbb{C}$, it is diagonalizable.

5. *(25 points)* Let $\mathbb{V}$ be a finite dimensional vector space over $\mathbb{C}$, and let $T \in \mathcal{L}(\mathbb{V}, \mathbb{V})$ be invertible. Prove that if $c \neq 0$ is an eigenvalue of T, then $1/c$ is an eigenvalue of T^{-1}.

Solution: If c is an eigenvalue of T with eigenvector α, we have

$$\alpha = T^{-1}(T\alpha) = T^{-1}(c\alpha) = cT^{-1}(\alpha)$$

Thus

$$\frac{1}{c}\alpha = T^{-1}(\alpha)$$

and so $1/c$ is an eigenvalue of T^{-1}.

The above proof works for any vector space $\mathbb{V}$. For a finite dimensional, complex vector space, we can give another, somewhat longer proof. Since it is also less enlightening, we'll only sketch the proof here. Since $\mathbb{V}$ is a complex vector space, there is a basis in which T corresponds to an upper-triangular matrix A, with the eigenvalues on the diagonal. In particular, c occurs at least once on the diagonal. The inverse of A is also upper-triangular, and since $A^{-1}A = I$, the diagonal of A^{-1} must have $1/c$ in the corresponding location. Hence $1/c$ is an eigenvalue of T^{-1}.