Solutions to Homework 9
MAT308, Spring 2011

Section 12.2, #36.

36. (a),(b) Let $\dot{x} = u$ and $\dot{y} = v$, so that $\ddot{x} = \dot{u}$ and $\ddot{y} = \dot{v}$. The equations of the given syatem then become

$$\begin{aligned}\dot{v} - 3x - 2y &= 0,\\ \dot{u} - y + 2x &= 0.\end{aligned}$$

The first-order system of dimension 4 equivalent to the given system is therefore

$$\begin{aligned}\dot{x} &= u,\\ \dot{y} &= v,\\ \dot{u} &= -2x + y,\\ \dot{v} &= 3x + 2y,\end{aligned}$$

which is the desired standard form.

(c) Write the original system in operator form:

$$\begin{aligned}-3x + (D^2 - 2)y &= 0,\\ (D^2 + 2)x - y &= 0.\end{aligned}$$

Operating on the second equation with $D^2 - 2$ and adding the result to the first equation eliminates y to get $-3x + (D^2 - 2)(D^2 + 2)x = 0$, or equivalently, $(D^4 - 7)x = 0$. The characteristic equation is $r^4 - 7 = (r^2 + \sqrt{7})(r^2 - \sqrt{7}) = 0$ with roots $r_{1,2} = \pm 7^{1/4}$, $r_{3,4} = \pm i7^{1/4}$. Therefore, with $\omega = 7^{1/4}$, the general solution for $x = x(t)$ is given by

$$x(t) = c_1 e^{\omega t} + c_2 e^{-\omega t} + c_3 \cos \omega t + c_4 \sin \omega t.$$

The second equation of the original system can then be solved for $y = y(t)$ to get

$$\begin{aligned}y(t) &= \ddot{x} + 2x\\ &= c_1\omega^2 e^{\omega t} + c_2\omega^2 e^{-\omega t} - c_3\omega^2 \cos \omega t - c_4\omega^2 \sin \omega t\\ &\qquad + 2(c_1 e^{\omega t} + c_2 e^{-\omega t} + c_3 \cos \omega t + c_4 \sin \omega t)\\ &= c_1(2 + \omega^2)e^{\omega t} + c_2(2 + \omega^2)e^{-\omega t} + c_3(2 - \omega^2)\cos \omega t + c_4(2 - \omega^2)\sin \omega t.\end{aligned}$$

Section 13.1, #7.

$$\begin{pmatrix} -e^t + 2e^{-t} \\ -2e^t + 2e^{-t} \end{pmatrix}$$

Section 13.1, #10.

10. First, identify the constant matrix A to be $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$. The eigenvalues λ of A satisfy

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) + 1 = \lambda^2 - 4\lambda + 5 = 0.$$

Hence, the eigenvalues are complex: $\lambda_{1,2} = 2 \pm i$.

For $\lambda_1 = 2 + i$, we want $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ to satisfy

$$(A - (2+i)I)\mathbf{u} = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -iu - v \\ u - iv \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This implies the scalar equations $-iu - v = 0$ and $u - iv = 0$. Since the second equation is i times the first equation, these equations are equivalent. Solving the first equation for v gives $v = -iu$. Choosing $u = 1$ gives the eigenvector $\mathbf{u} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

For $\lambda_2 = 2 - i$, we want $\mathbf{v} = \begin{pmatrix} u \\ v \end{pmatrix}$ to satisfy

$$(A - (2-i)I)\mathbf{v} = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} iu - v \\ u + iv \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This implies the scalar equations $iu - v = 0$ and $u + iv = 0$. Since the first equation is i times the second equation, these equations are equivalent, Solving the first equation for v gives $v = iu$, so that choosing $u = 1$ yields the eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$. Therefore, with $\mathbf{x}(t) = (x(t), y(t))$, the general solution of the given system is

$$\begin{aligned}
\mathbf{x}(t) &= c_1 e^{\lambda_1 t}\mathbf{u} + c_2 e^{\lambda_2 t}\mathbf{v} = c_1 e^{(2+i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 e^{(2-i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} \\
&= c_1 e^{2t}(\cos t + i\sin t)\begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 e^{2t}(\cos t - i\sin t)\begin{pmatrix} 1 \\ i \end{pmatrix} \\
&= e^{2t}\left[\begin{pmatrix} c_1\cos t + ic_1\sin t \\ -ic_1\cos t + c_1\sin t \end{pmatrix} + \begin{pmatrix} c_2\cos t - ic_2\sin t \\ ic_2\cos t + c_2\sin t \end{pmatrix}\right] \\
&= e^{2t}\begin{pmatrix} (c_1 + c_2)\cos t + i(c_1 - c_2)\sin t \\ -i(c_1 - c_2)\cos t + (c_1 + c_2)\sin t \end{pmatrix}.
\end{aligned}$$

The initial condition gives $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{x}(0) = \begin{pmatrix} c_1 + c_2 \\ -i(c_1 - c_2) \end{pmatrix}$, which implies the scalar system $c_1 + c_2 = 1$, $-i(c_1 - c_2) = 0$. The second equation implies $c_1 = c_2$, so that the first equation gives $c_1 = c_2 = \frac{1}{2}$. The desired solution is therefore

$$\mathbf{x}(t) = e^{2t}\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} e^{2t}\cos t \\ e^{2t}\sin t \end{pmatrix}.$$

Section 13.1, #11.

(a) $x_h(t) = ae^{-2t}\begin{pmatrix}1\\-1\\-1\end{pmatrix} + be^t\begin{pmatrix}-1\\4\\1\end{pmatrix} + ce^{3t}\begin{pmatrix}1\\2\\1\end{pmatrix}$

(b) $x_p(t) = \begin{pmatrix}-5/2\\9/2\\3/2\end{pmatrix}$

(c) $x(t) = -\frac{1}{2}e^{-2t}\begin{pmatrix}1\\-1\\-1\end{pmatrix} - 2e^t\begin{pmatrix}-1\\4\\1\end{pmatrix} + 2e^{3t}\begin{pmatrix}1\\2\\1\end{pmatrix} + \begin{pmatrix}-5/2\\9/2\\3/2\end{pmatrix}$

Section 13.2ABC, #2.

2. Since $A = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ and $A^2 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = I$, it follows that all odd powers of A are equal to A and all even powers of A are equal to the identity matrix. Hence,

$$e^{tA} = \sum_{k=0}^{\infty}\frac{t^k}{k!}A^k = \sum_{j=0}^{\infty}\frac{t^{2j}}{(2j)!}I + \sum_{j=0}^{\infty}\frac{t^{2k+1}}{(2k+1)!}A.$$

We recognize the first series on the right as $\cosh t$ and the second series on the right as $\sinh t$. Hence,

$$e^{tA} = (\cosh t)I + (\sinh t)A = \begin{pmatrix}\cosh t & 0\\0 & \cosh t\end{pmatrix} + \begin{pmatrix}0 & \sinh t\\\sinh t & 0\end{pmatrix} = \begin{pmatrix}\cosh t & \sinh t\\\sinh t & \cosh t\end{pmatrix}.$$

Section 13.2ABC, #11.

$$e^{tA} = \begin{pmatrix}-e^t + 2e^{-t} & e^t - e^{-t}\\2e^t + 2e^{-t} & 2e^t - e^{-t}\end{pmatrix}$$

Section 13.2ABC, #22.

22. For any square matrix A with real entries, each term of the series $\sum_{k=0}^{\infty}(i)^k\frac{t^k}{k!}A^k$ is either a matrix with real entries (when the exponent is even) or i times a matrix with real entries (when the exponent is odd). We can conclude that $\cos tA$ and $\sin tA$ have the series representations

$$\cos tA = \sum_{j=0}^{\infty}(-1)^j\frac{t^{2j}}{(2j)!}A^{2j} \qquad \text{and} \qquad \sin tA = \sum_{j=0}^{\infty}(-1)^j\frac{t^{2j+1}}{(2j+1)!}A^{2j+1}.$$

Section 13.2ABC, #22 (continued)

In the special case $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, use mathematical induction to deduce that $A^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ for all integers k. The series definition of $\cos tA$ can be explicitly written as

$$\cos tA = \sum_{j=0}^{\infty}(-1)^j \frac{t^{2j}}{(2j)!}\begin{pmatrix} 1 & 2j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sum_0^\infty (-1)^j \frac{t^{2j}}{(2j)!} & \sum_1^\infty (-1)^j \frac{t^{2j}}{(2j-1)!} \\ 0 & \sum_0^\infty (-1)^j \frac{t^{2j}}{(2j)!} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_0^\infty (-1)^j \frac{t^{2j}}{(2j)!} & -t\sum_0^\infty (-1)^j \frac{t^{2j+1}}{(2j+1)!} \\ 0 & \sum_0^\infty (-1)^j \frac{t^{2j}}{(2j)!} \end{pmatrix} = \begin{pmatrix} \cos t & -t\sin t \\ 0 & \cos t \end{pmatrix}.$$

A similar manipulation gives $\sin tA = \begin{pmatrix} \sin t & t\cos t \\ 0 & \sin t \end{pmatrix}$. Hence,

$$\cos(-tA) = \cos(-t)A = \begin{pmatrix} \cos(-t) & -(-t)\sin(-t) \\ 0 & \cos(-t) \end{pmatrix} = \begin{pmatrix} \cos t & -t\sin t \\ 0 & \cos t \end{pmatrix} = \cos tA,$$

$$\sin(-tA) = \sin(-t)A = \begin{pmatrix} \sin(-t) & (-t)\cos(-t) \\ 0 & \sin(-t) \end{pmatrix} = -\begin{pmatrix} \sin t & t\cos t \\ 0 & \sin t \end{pmatrix} = -\sin tA.$$

This proves the relations given in part (a) of Exercise 21. To prove the relations given in part (b) of Exercise 21, we first compute

$$\frac{d}{dt}\cos tA = \frac{d}{dt}\begin{pmatrix} \cos t & -t\sin t \\ 0 & \cos t \end{pmatrix} = \begin{pmatrix} -\sin t & -t\cos t - \sin t \\ 0 & \cos t \end{pmatrix}$$

$$\frac{d}{dt}\sin tA = \frac{d}{dt}\begin{pmatrix} \sin t & t\cos t \\ 0 & \sin t \end{pmatrix} = \begin{pmatrix} \cos t & -t\sin t + \cos t \\ 0 & \cos t \end{pmatrix}.$$

We also have

$$-A\sin tA = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \sin t & t\cos t \\ 0 & \sin t \end{pmatrix} = \begin{pmatrix} -\sin t & -t\cos t - \sin t \\ 0 & -\sin t \end{pmatrix},$$

$$A\cos tA = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos t & -t\sin t \\ 0 & \cos t \end{pmatrix} = \begin{pmatrix} \cos t & -t\sin t + \cos t \\ 0 & \cos t \end{pmatrix}.$$

Comparing the two groups of equations, we see that

$$\frac{d}{dt}\cos tA = -A\sin tA \qquad \text{and} \qquad \frac{d}{dt}\sin tA = A\cos tA.$$

This proves the relations given in part (b) of Exercise 21. Finally,

$$(\cos tA)^2 + (\sin tA)^2 = \begin{pmatrix} \cos t & -t\sin t \\ 0 & \cos t \end{pmatrix}^2 + \begin{pmatrix} \sin t & t\cos t \\ 0 & \sin t \end{pmatrix}^2$$

$$= \begin{pmatrix} \cos^2 t & -2t\sin t\cos t \\ 0 & \cos^2 t \end{pmatrix} + \begin{pmatrix} \sin^2 t & 2t\cos t\sin t \\ 0 & \sin^2 t \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2 t + \sin^2 t & 0 \\ 0 & \cos^2 t + \sin^2 t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Section 13.2D, #7.

$$e^{3t}\begin{pmatrix}1-t & 0 & t\\ -t & 1 & t\\ -t & 0 & 1+t\end{pmatrix}$$

Section 13.2D, #16.

16. If A is the given matrix then

$$\det A=\begin{vmatrix}1 & 2 & 3\\ -1 & 1 & 0\\ 0 & 3 & 3\end{vmatrix}=\begin{vmatrix}1 & 0\\ 3 & 3\end{vmatrix}-(-1)\begin{vmatrix}2 & 3\\ 3 & 3\end{vmatrix}=(3-0)+(6-9)=0.$$

Hence, A^{-1} does not exist.

18. If A is the given matrix then

$$\det A=\begin{vmatrix}t & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{vmatrix}=2t.$$

Thus, A^{-1} exists for $t\neq 0$. Moreover, the characteristic polynomial of A is

$$P(\lambda)=\det(A-\lambda I)=\begin{vmatrix}t-\lambda & 0 & 0\\ 0 & 2-\lambda & 0\\ 0 & 0 & 1-\lambda\end{vmatrix}=(t-\lambda)(2-\lambda)(1-\lambda)$$
$$=-\lambda^3+(3+t)\lambda^2-(3t+2)\lambda+2t.$$

By the Cayley-Hamilton Theorem, $P(A)=0$, so that $-A^3+(3+t)A^2-(3t+2)A+2tI=0$. If $t\neq 0$, multiply this equation by A^{-1} to get $-A^2+(3+t)A-(3t+2)I+2tA^{-1}=0$. Solving for A^{-1} gives $A^{-1}=(1/2t)\big(A^2-(3+t)A+(3t+2)I\big)$. Hence,

$$A^{-1}=\frac{1}{2t}\left[\begin{pmatrix}t^2 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\end{pmatrix}-(3+t)\begin{pmatrix}t & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{pmatrix}+(3t+2)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right]$$
$$=\frac{1}{2t}\begin{pmatrix}2 & 0 & 0\\ 0 & t & 0\\ 0 & 0 & 2t\end{pmatrix}=\begin{pmatrix}1/t & 0 & 0\\ 0 & 1/2 & 0\\ 0 & 0 & 1\end{pmatrix}.$$