MAT 307 FALL 2011 MIDTERM I EXAM

Name: SOLUTIONS **ID Number:**

Problem	1	2	3	4	5	6	Total
Score							

Directions: Each problem is 15 points; some problems are easier, some harder. Do all of your work on these pages and cross out any work that should be ignored. Use the backs of pages if needed, indicating where you've done this. Notes, books, calculators and discussions with others are not permitted.

1. Find the equation of the plane in $\mathbb{R}^3$ through the 3 points $(1,0,0)$, $(2,1,1)$, $(1,2,3)$.

LET $A=(1,0,0)$, $B=(2,1,1)$ $C=(1,2,3)$.

THEN THE VECTORS $\overrightarrow{B-A} = \langle 1,1,1 \rangle$

AND $\overrightarrow{C-A} = \langle 0,2,3 \rangle$

ARE IN THE PLANE,

SO ~~THE~~ NORMAL IS

$$\overrightarrow{BA} \times \overrightarrow{CA} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{vmatrix} = (3-2)\vec{i} - (3-0)\vec{j} + (2-0)\vec{k}$$

$$= \langle 1, -3, 2 \rangle$$

SINCE THE POINT $(1,0,0)$ LIES IN THE PLANE, EQUATION IS

$$(x-1) - 3y + 2z = 0.$$

ALTERNATIVELY, IN PARAMETRIC FORM, THE PLANE IS

$$\vec{R}(t,s) = \langle 1,0,0 \rangle + t\langle 1,1,1 \rangle + s\langle 0,2,3 \rangle$$

$$= \langle t+1,\ t+2s,\ t+3s \rangle.$$

2. Consider the linear system

$$\begin{aligned} x + y + z &= 1 \\ x - y &= 1 \\ x + y &= 2 \end{aligned}$$

a. Write this system in matrix form $A \cdot \mathbf{x} = \mathbf{b}$.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

b. Find the row-reduced form of this linear system

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 0 & 2 \end{array}\right) \xrightarrow{\text{SUBTRACT ROW 1 FROM EACH}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right)$$

CHG SIGN OF LAST ROW
ADD / SUBTRACT

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow[\text{SUB TO 1ST}]{\text{DIVIDE MIDDLE ROW BY -2}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

c. Find all solutions of the linear system and describe the solution set.

ONLY ONE SOLUTION, THE POINT

$x = 3/2$

$y = 1/2$

$z = -1$

3.

a. Find all real numbers a for which the matrix $\begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & a \\ 1 & a & 1 \end{pmatrix}$ has an inverse and does not have an inverse.

DET IS $1\cdot(a-a^2)+1(2-a) = -a^2-a+2$:
$= -(a+2)(a-1)$

THIS IS ZERO IF

$a=-2$ OR $a=1$.

THUS, MATRIX HAS AN INVERSE IF $a \neq -2$ AND $a \neq 1$

NO INVERSE IF $a=2$ OR $a=1$.

b. Find all solutions of the linear system and describe the structure of the solution set:

$$\begin{aligned} (1)\quad x-y&=-1,\\ (2)\quad 2x+z&=2\\ (3)\quad x+y+z&=1. \end{aligned}$$

NOTE THAT (2) − (1) GIVES $x+y+z=3$,
WHICH IS INCONSISTENT WITH
(3) $x+y+z=1$,

THUS THERE ARE NO SOLUTIONS

THE SOLUTION SET IS
THE EMPTY SET.

4. The function

$$c(t) = (e^t, e^{-t}, t)$$

is a curve in $\mathbb{R}^3$.

a. Compute the velocity vector of $c(t)$ at any t and find the equation of the tangent line to c at $t = 1$.

$\vec{c}\,'(t) = \frac{d}{dt}\vec{c}(t) = \langle e^t, -e^{-t}, 1 \rangle$

$\vec{c}\,'(1) = \langle e, -\frac{1}{e}, 1 \rangle$ IS TANGENT TO $c(t)$ AT $t=1$,

SO THE TANGENT LINE IN PARAMETRIC FORM IS $\vec{c}(1) + t\,\vec{c}\,'(1)$, ie

$$T(t) = \langle e, \tfrac{1}{e}, 1 \rangle + t\langle e, \tfrac{-1}{e}, 1 \rangle$$

OR
$$\begin{cases} x = e(1+t) \\ y = \frac{1}{e}(1-t) \\ z = 1 \end{cases}$$

b. What is the shortest distance of any point on this curve to the origin $(0, 0, 0)$. (Hint: Minimize the function $|c(t)|^2$ as a function of t).

LET $R(t) = |c(t)|^2 = e^{2t} + e^{-2t} + t^2$

TO FIND CRITICAL POINTS,

$$R'(t) = 2e^{2t} - 2e^{-2t} + 2t.$$

BY OBSERVATION, $R'(0) = 0$, SO THIS IS A CRITICAL POINT.

FURTHER, $R''(t) = 4e^{2t} + 4e^{-2t} + 2$, WHICH IS ALWAYS POSITIVE, SO THIS IS A LOCAL MIN (AND THE ONLY MINIMUM.)

SO, THE MIN IS AT $t = 0$, ie AT $c(0) = \langle 1, 1, 0 \rangle$

IN THIS CASE, THE DISTANCE TO THE ORIGIN IS

$\sqrt{2}$,

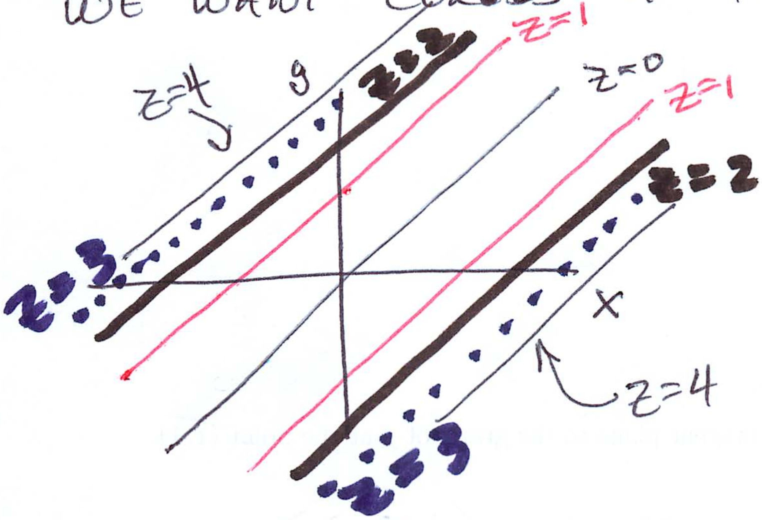
5.

a. Sketch the level curves of the function

$$f(x,y) = (x-y)^2$$

in the plane $\mathbb{R}^2$. Label your curves by the values of f on them, and choose at least 4 values of f to get a good picture of the behavior of the level curves over the full domain of this function.

WE WANT CURVES OF FORM $(x-y)^2 = \text{CONST.}$ FOR VARIOUS CONSTANTS

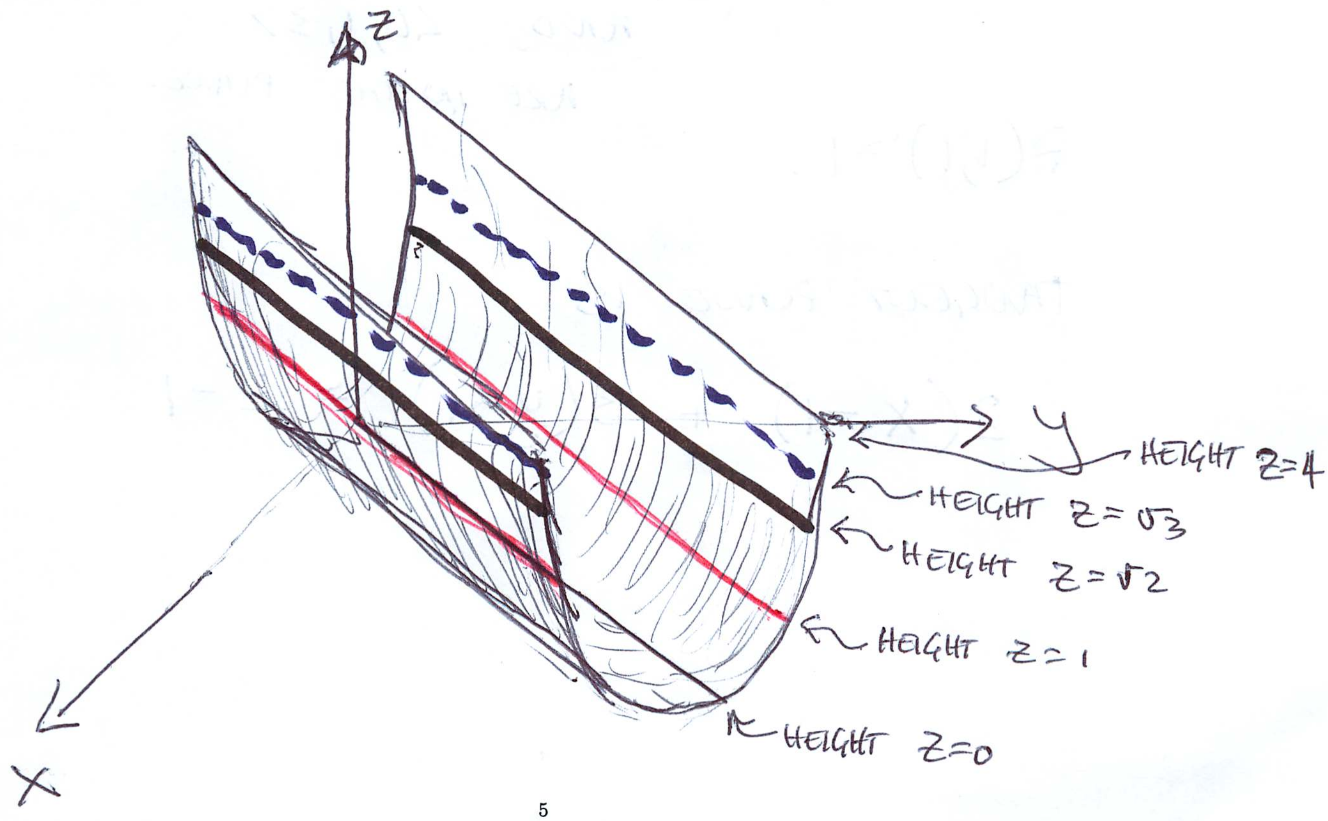


EACH OF THESE IS OF THE FORM

$$x - y = \pm\sqrt{c}$$

(ie, A PAIR OF LINES OF SLOPE 1

b. Sketch the graph of f in $\mathbb{R}^3$.



6. Consider the function

$$f(x,y) = x^2y^3.$$

a. Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at any (x,y).

$$\frac{\partial f}{\partial x} = 2xy^3$$

$$\frac{\partial f}{\partial y} = 3x^2y^2$$

b. Find the equation of the tangent plane to the graph of f at the point $(1,1)$.

AT $(1,1)$, THE VECTORS $\langle 1,0,2\rangle$ AND $\langle 0,1,3\rangle$ ARE IN THE PLANE.

$f(1,1) = 1$.

TANGENT PLANE IS

$$2(x-1) + 3(y-1) = z-1$$